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Perturbation from an elliptic Hamiltonian of degree four—III global centre

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Abstract

The paper deals with Liénard equations of the form $\dot{x} = y$, $\dot{y} = P(x) + yQ(x)$ with P and Q polynomials of degree, respectively, 3 and 2. Attention goes to perturbations of the Hamiltonian vector fields with an elliptic Hamiltonian of degree four, exhibiting a global centre. It is proven that the least upper bound of the number of zeros of the related elliptic integral is four, and this is a sharp one.

This result permits to prove the existence of Liénard equations of type (3,2) with a quadruple limit cycle, with both a triple and a simple limit cycle, with two semistable limit cycles, with one semistable and two simple limit cycles or with four simple limit cycles.

0. Introduction

Like [DL1,DL2] this paper deals with elliptic integrals that are obtained by integrating the 1-forms $y(\alpha + \beta x + x^2)\,dx$ over the level curves of the Hamiltonians $H(x,y) = \frac{y^2}{2} \pm \frac{x^4}{4} + \frac{ax^3}{3} + \frac{bx^2}{2}$.

In [DL1] we gave a general introduction to the subject describing some problems where the setting naturally shows up. We also made a complete study of the saddle loop and the two saddle cycle cases. In [DL2] we dealt with the cuspidal loop case.

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In this paper, we want to study the case of a global centre (case (C) as presented in [DL1]). Up to a linear coordinate change there is a 1-parameter family $H_\lambda(x,y) = \frac{y^2}{2} + \frac{1}{4}x^4 - \frac{2\lambda}{3}x^3 + \frac{1}{2}x^2$ of Hamiltonians, with $\lambda \in [0,1)$, representing a global centre. For $\lambda \sim 1$, the family tends in a regular way to the cuspidal loop case, treated in [DL2]. Recall that for the perturbation from a cuspidal loop the sharp upper bound for the number of zeros of the related elliptic integrals is four. If restricting to level curves "inside" or "outside" the cuspidal loop, then the sharp upper bound is, respectively, 2 and 3. There is no possibility to exhibit quadruple limit cycles. We will prove in this paper that for $\lambda < 1$, but close to 1, the corresponding sharp upper bound is also four, and we give a formal proof that in Liénard equations of type $(3,2)$ one can encounter quadruple limit cycles, as was conjectured in [KKR]. The quadruple limit cycles, whose existence we prove, occur for vector fields $\dot{x} = y, \dot{y} = -x(x^2 - 2\lambda x + 1) + \delta(\alpha + \beta x + x^2)y$ with some $\lambda \in (0,1)$, $\alpha > 0$, $\beta < 0$, and $\delta > 0$ as small as wanted. In theorem A of Section 1 we prove that, for all $\lambda \in (0,1)$, four is an absolute upper bound for the number of zeros of the Abelian integrals and we also describe precisely the bifurcation diagram of the zeros for $\lambda \sim 0$ and $\lambda \sim 1$.

1. Formulation of the problem and main results

We consider a general form of an elliptic Hamiltonian function of degree four

$$H(x,y) = \frac{y^2}{2} + \frac{a}{4}x^4 + \frac{b}{3}x^3 + \frac{c}{2}x^2 \tag{1}$$

with $a \neq 0$. The corresponding Hamiltonian system is

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x(ax^2 + bx + c). \end{cases} \tag{2}$$

If (2) has a non-degenerate global centre at the origin, then $c > 0$ and $b^2 - 4ac < 0$, hence $a > 0$. By the transformation $(x,y,t) \mapsto (\sqrt{\frac{c}{a}}x, \frac{c}{\sqrt{a}}y, \frac{1}{\sqrt{c}}t)$ system (2) keeps the same form with $a = c = 1$ and $b^2 < 4$. If $b \geqslant 0$, then by the change of coordinates $(x,y) \mapsto (-x,-y)$ system (2) still has the same form with $b \leqslant 0$. Thus, without loss of generality, we will take $a = c = 1$ and $b = -2\lambda$ with $\lambda \in [0,1)$. If (2) has a degenerate global centre at the origin then $c = b = 0$, a can be changed to 1. The study for this case is simple, see Remark 2.5.

Now we consider a perturbation from the global centre

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x(x^2 - 2\lambda x + 1) + \delta(\alpha + \beta x + x^2)y, \end{cases} \tag{3_δ}$$

with $\lambda \in [0,1), \delta$ strictly positive but small, α and β are constants.

The Hamiltonian system $(3)_0$ has as first integral

$$H(x,y) = \frac{y^2}{2} + \frac{1}{4}x^4 - \frac{2\lambda}{3}x^3 + \frac{1}{2}x^2. \tag{4}$$

All orbits of $(3)_0$ for $\lambda \in [0,1)$ are closed, surrounding the centre $(0,0)$; it is symmetric for $\lambda = 0$. If $\lambda = 1$, $(3)_0$ has a cuspidal loop.

Related to $(3)_0$, we consider the Abelian integral

$$I(h) = \int_{\Gamma_h} (\alpha + \beta x + x^2) y\, dx = \alpha I_0(h) + \beta I_1(h) + I_2(h), \tag{5}$$

where Γ_h is the level curve $\{(x,y) \mid H(x,y) = h, h > 0\}$, oriented clockwise; and $I_k(h) = \int_{\Gamma_h} x^k y\, dx,\ k = 0,1,2.$

The main result in this paper is the following

Theorem A. *If we integrate the* 1*-forms* $(x^2 + \beta x + \alpha) y\, dx$ *over the compact level curves* $\Gamma_{h,\lambda} = H_\lambda^{-1}(h)$, *with* $h \in (0, \infty)$ *and* $\lambda \in (0,1)$, *of the Hamiltonians*

$$H(x,y) = \frac{y^2}{2} + \frac{1}{4}x^4 - \frac{2\lambda}{3}x^3 + \frac{1}{2}x^2$$

then:

(1) *for all* λ, *and for all constants* α *and* β, *the maximum number of zeros is four, taking into account the multiplicity.*
(2) *there exists some* $\lambda^* \in (0,1)$ *at which a quadruple zero shows up*, *occurring in a complete swallowtail-bifurcation of zeros* (*depending on the parameters* (α, β, λ)).
(3) *for* $\lambda \sim 0$ (*resp.*, $\lambda \sim 1$) *the bifurcation diagram of the zeros is as represented in* Fig. 1a (*resp.*, 1b), *the digit indicating the number of zeros in the different open regions. In these bifurcation diagrams* $H = \{\alpha = 0\}$ *stands for a line of zeros at* $h = 0$, *representing Hopf bifurcations*, *and the other curves represent double zeros. The points* $H_2 = \{(0, -\frac{1}{2\lambda})\}$ *and* T_1, T_2 *represent*, *respectively*, *a double zero at* $h = 0$ *and triple zeros on* $(0, \infty)$.

Conjecture. *There is only one value* $\lambda^* \in (0,1)$ *at which a quadruple zero shows up*; *for* $\lambda \in (0, \lambda^*)$ (*resp.*, $\lambda \in (\lambda^*, 1)$) *the bifurcation diagram of the zeros is as represented in Fig.* 1a (*resp.*, 1b).

Since $I_0(h) > 0$ for $h > 0$, instead of (5), we also consider

$$\widetilde{I}(h) = \alpha + \beta P(h) + Q(h), \tag{6}$$

where

$$P(h) = \frac{I_1(h)}{I_0(h)}, \qquad Q(h) = \frac{I_2(h)}{I_0(h)}. \tag{7}$$

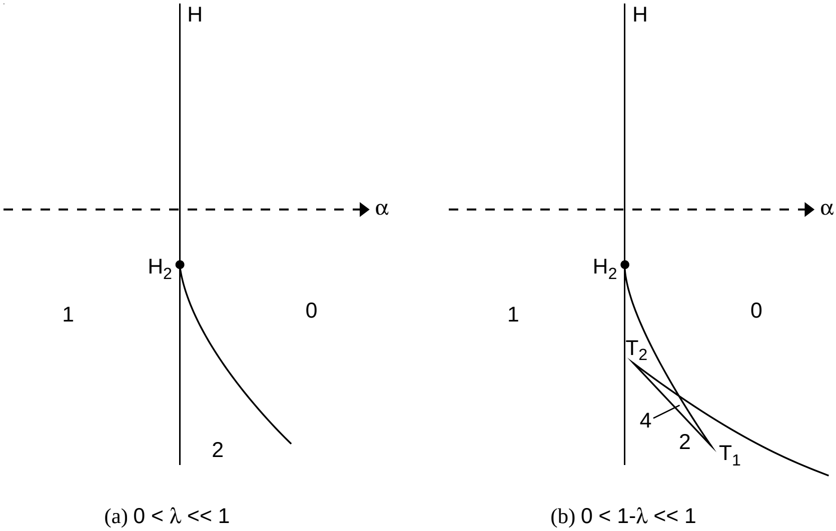

Fig. 1. Bifurcation diagram of zeros.

It is easy to see that $\lim_{h\to 0+} P(h) = \lim_{h\to 0+} Q(h) = 0$, so we define $P(0) = Q(0) = 0$. Let

$$\Sigma_\lambda = \{(P, Q)(h) \mid h \in [0, +\infty)\}. \tag{8}$$

Note that the function $H(x, y)$, and hence also Γ_h, $P(h)$ and $Q(h)$ depend on λ; Σ_λ is a family of curves in (P, Q)-plane depending on the parameter λ; for any fixed $\lambda \in [0, 1]$, Σ_λ is parametrized by $h \in [0, +\infty)$.

It is clear that for fixed α and β the number of zeros of the Abelian integral (5) for $h > 0$ is equal to the number of intersection points of the straight line

$$\mathscr{L}_{\alpha,\beta} : \alpha + \beta P + Q = 0, \tag{9}$$

with the curve Σ_λ in (P, Q)-plane.

It is not difficult to see that Σ_λ is C^∞ in $h \in [0, +\infty)$ for $\lambda \in [0, 1)$, and Σ_1 is C^1 in $h \in [0, +\infty)$ and C^∞ in $h \in [0, \frac{1}{12}) \cup (\frac{1}{12}, +\infty)$, where $h = \frac{1}{12}$ corresponds to the cuspidal loop. One way to check the C^∞ property near $h = 0$ is to consider Eq. (36).

Definition. A point on Σ_λ is called triple (resp., quadruple, or higher than quadruple), if at this point the curve Σ_λ and its tangent line have a contact which is exactly triple (resp., exactly quadruple, or more than quadruple).

Note that if $\lambda = 0$ then $I_1(h) \equiv 0$ (hence $P(h) \equiv 0$ and Σ_λ becomes a semi-straight line). We will prove that $Q'(h) > 0$ for $\lambda \in [0, 1]$ (Lemma 2.4), therefore (6) and the Abelian integral (5) have at most one zero for $\lambda = 0$ and $h > 0$, taking into account the multiplicity. Thus, we will only consider the case $\lambda > 0$.

Theorem A immediately follows from the next result.

Theorem B. *The family of curves Σ_λ, as defined in* (8), *has the following properties*:

(B1) *For any $\lambda \in (0,1)$ and any constants α and β, the curve $\sum_\lambda$ and the straight line $\mathscr{L}_{\alpha,\beta}$ have no more than 4 intersection points, taking into account the multiplicity. Hence $\sum_\lambda$ has no point which is higher than quadruple.*

(B2) *There exists a constant $\sigma_1 \in (0,1)$, such that for $\lambda \in (0, \sigma_1]$ the curve $\sum_\lambda$ has no triple nor higher than triple point.*

(B3) *There is a $h_1 > 0$, such that for all $\lambda \in [\sigma_1, 1)$ $\sum_\lambda$ has no triple nor higher than triple point for $0 \leqslant h \leqslant h_1$.*

(B4) *There is a $h_2 > h_1$, such that for all $\lambda \in [\sigma_1, 1)$, $\sum_\lambda$ has no triple nor higher than triple point for $h \geqslant h_2$.*

(B5) *There exists a $\sigma_2 \in (0,1)$, such that for all $\lambda \in [1-\sigma_2, 1)$ $\sum_\lambda$ has exactly two triple points, and has no quadruple nor higher than quadruple point.*

As a consequence, from Theorem B, we have

Theorem C. *There is a $\lambda^* \in (0,1)$ such that Σ_{λ^*} has a quadruple point, which is a coalescence of two triple points of Σ_λ as $\lambda \to \lambda^*$ with $\lambda > \lambda^*$.*

Note that $(3)_\delta$ is a cubic Liénard equation with (small) quadratic damping. Theorems B and C imply the following result.

Theorem D. *There exist constants α, β and $\lambda \in (0,1)$ such that system $(3)_\delta$, with δ small, has a quadruple limit cycle, which, for suitable small changes of λ, α and β, can be splitted into either four simple limit cycles, two simple and one semi-stable limit cycles, two semi-stable limit cycles, or one simple and one triple limit cycle.*

We will prove the conclusions (B1)–(B5) of Theorem B in Sections 4–8. The proof goes along the lines explained in [DL1] and also used in [DL2]. In Section 2 we directly study some properties of $P(h)$ and $Q(h)$.

In Section 3 we study some properties of $\omega(h)$ and $v(h)$, where $\omega(h) = \frac{I_1''(h)}{I_0''(h)}$ and $v(h) = \frac{I_2''(h)}{I_0''(h)}$. Finally, in the subsequent sections we combine both results in making a detailed analysis of the shape of the curve $(P(h), Q(h))$ in (P, Q)-plane; this leads to a proof of the respective conclusions (B1)–(B5) in the respective Sections 4–8.

2. Monotonicity of $P(h)$, $Q(h)$ and $\frac{Q(h)}{P(h)}$

Since $\lambda \in [0,1]$, along Γ_h we have $x\dot{y} = -x^2(x^2 - 2\lambda x + 1) < 0$ for $x \neq 0$ (keeping also $x \neq 1$ if $\lambda = 1$). Hence for any $x \in [A(h), 0]$ there exists a unique $\widetilde{x} \in [0, B(h)]$, such

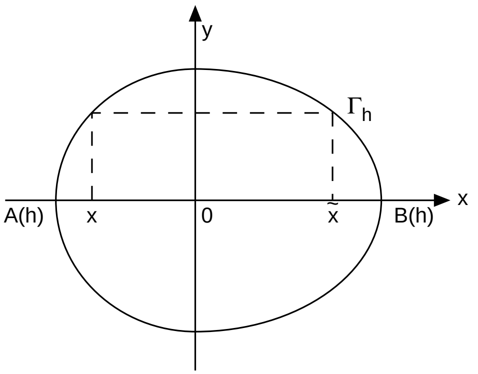

Fig. 2. Introducing $\widetilde{x}(x)$.

that $H(x,y) = H(\widetilde{x}, y)$, where $A(h)$ and $B(h)$ are the abscissa of the intersection points of Γ_h with the x-axis, see Fig. 2.

We define $u = u(x)$ and $v = v(x)$ as follows:

$$u = x + \widetilde{x}, \qquad v = x\widetilde{x}, \tag{10}$$

where $\widetilde{x} = \widetilde{x}(x)$ is defined as above. For $\lambda = 0$ we have $\widetilde{x} = -x$ and $B(h) = -A(h)$, while for $\lambda \in (0, 1]$ we can prove the following:

Lemma 2.1. *For $\lambda \in (0, 1]$ and $x < 0$ we have*

(i) $0 < u < \frac{4\lambda}{3}$ *and* $\frac{du}{dx} < 0$.

(ii) $A(h) + B(h) = \frac{4\lambda}{3} - \frac{2\lambda(9-8\lambda^2)}{27}\frac{1}{\sqrt{h}} + o(\frac{1}{\sqrt{h}})$ *as* $h \to +\infty$.

Proof. Let $\Phi(x) = H(x, 0) = \frac{x^4}{4} - \frac{2\lambda}{3}x^3 + \frac{x^2}{2}$, then from $\Phi(x) = \Phi(\widetilde{x})$ and (10) we obtain

$$(3u^2 - 8\lambda u + 6)u = 2(3u - 4\lambda)v. \tag{11}$$

Since $v < 0$ and $|u| \ll 1$ for $0 < |x| \ll 1$, from (11) we have $u > 0$ for $0 < |x| \ll 1$. We claim that $u > 0$ for all $x \in [A(h), 0)$. In fact, if there is a $x < 0$ such that $u = 0$, then from (11) we get $v = 0$ which is impossible. We also note that $3u^2 - 8\lambda u + 6 > 0$ by using $0 \leqslant \lambda \leqslant 1$, hence from (11) we obtain

$$0 < u < \frac{4}{3}\lambda, \qquad v = \frac{(3u^2 - 8\lambda u + 6)u}{2(3u - 4\lambda)}. \tag{12}$$

Next, by (10) we have for $x < 0$

$$\frac{du}{dx} = 1 + \frac{d\widetilde{x}}{dx} = 1 + \frac{\Phi'(x)}{\Phi'(\widetilde{x})} = \frac{1}{\Phi'(\widetilde{x})}(\Phi'(x) + \Phi'(\widetilde{x})). \tag{13}$$

It is easy to see that

$$\Phi'(x) + \Phi'(\tilde{x}) = x^3 + \tilde{x}^3 - 2\lambda(x^2 + \tilde{x}^2) + (x + \tilde{x}). \tag{14}$$

Using

$$\begin{cases} x^3 + \tilde{x}^3 = u^3 - 3uv, \\ x^2 + \tilde{x}^2 = u^2 - 2v, \end{cases} \tag{15}$$

as well as (12), we obtain from (14) that

$$\Phi'(x) + \Phi'(\tilde{x}) = -\frac{u}{2}(u^2 - 4\lambda u + 4) < 0.$$

On the other hand, $\Phi'(\tilde{x}) = \tilde{x}(\tilde{x}^2 - 2\lambda\tilde{x} + 1) > 0$ since $\tilde{x} > 0$, hence $\frac{du}{dx} < 0$ by (13). Conclusion (i) is proved.

For a particular case, we take $x = A(h)$, then by conclusion (i) $0 < u = A(h) + B(h) < \frac{4\lambda}{3}$. On the other hand, it is easy to see that $v = A(h) \cdot B(h) = -2\sqrt{h} + O(1)$ as $h \to +\infty$. Hence, from (11) we must have $u = \frac{4\lambda}{3} + o(1)$ as $h \to +\infty$. Substituting these asymptotic expressions into the adapted form of (11):

$$u = \frac{4}{3}\lambda + \frac{(3u^2 - 8\lambda u + 6)u}{6v},$$

we obtain immediately conclusion (ii). □

Lemma 2.2. $P'(h) > 0$ *for* $0 < h < +\infty$, $0 < \lambda \leqslant 1$.

Proof. We use a technique which first appeared in [L]. Let

$$s(h) = \frac{A(h) + B(h)}{2}, \qquad r(h) = \frac{B(h) - A(h)}{2}, \tag{16}$$

where $A(h) < 0 < B(h)$ are as in Lemma 2.1 (see Fig. 2). Using definition (7) and the fact that $I_0(h) > 0$ for $h > 0$, we know that the sign of $P'(h)$ for $h > 0$ is the same as for the following expression:

$$\begin{aligned} & I_1'(h)I_0(h) - I_0'(h)I_1(h) \\ &\quad = \int_{A(h)}^{B(h)} \frac{x}{y(x)}\,dx \int_{A(h)}^{B(h)} y(x)\,dx - \int_{A(h)}^{B(h)} \frac{1}{y(x)}\,dx \int_{A(h)}^{B(h)} xy(x)\,dx \\ &\quad = \int_{A(h)}^{B(h)} \frac{x - s(h)}{y(x)}\,dx \int_{A(h)}^{B(h)} y(x)\,dx - \int_{A(h)}^{B(h)} (x - s(h))y(x)\,dx \int_{A(h)}^{B(h)} \frac{1}{y(x)}\,dx, \end{aligned} \tag{17}$$

where $y(x) \geqslant 0$ is defined by $H(x,y) = h$. By the change of variable $x = t + s(h)$, we have

$$\int_{A(h)}^{B(h)} (x - s(h)) y^{\pm 1}(x)\, dx = \int_{-r(h)}^{r(h)} t y^{\pm 1}(s(h) + t)\, dt$$
$$= \int_{0}^{r(h)} t[y^{\pm 1}(s(h) + t) - y^{\pm 1}(s(h) - t)]\, dt, \qquad (18)$$

where $r(h) > 0$ is given in (16). Hence, if for any $h > 0$

$$y(s(h) + t) - y(s(h) - t) \leqslant 0 \quad \text{for } t \in [0, r(h)] \qquad (19)$$

is satisfied, and the equality is not identical, then by using (18) we conclude immediately that (17) is positive and the proof of the lemma is finished.

Recall that $y = y(x) \geqslant 0$ is defined by $H(x,y) = \frac{y^2}{2} + \Phi(x) = h$, hence

$$y^2(s(h) + t) - y^2(s(h) - t) = 2\Psi(t), \qquad (20)$$

where $\Psi(t) = \Phi(s(h) - t) - \Phi(s(h) + t)$. Note that $\Phi(x)$ is a polynomial of x of degree 4, hence $\Psi(t)$ is a polynomial of t of degree 3. Obviously, $t = 0$ is a root of $\Psi(t)$; and $t = \pm r(h)$ are two more roots of $\Psi(t)$ since $A(h)$ and $B(h)$ are roots of $\Phi(x)$, and $s(h)$ and $r(h)$ are defined by $A(h)$ and $B(h)$ in (16). Thus

$$\Psi(t) = kt(t - r(h))(t + r(h)), \qquad (21)$$

where the coefficient k is easy to find:

$$k = \frac{4}{3}\lambda - 2s(h) = \frac{4\lambda}{3} - (A(h) + B(h)). \qquad (22)$$

By Lemma 2.1 $k > 0$ if $\lambda > 0$, hence (19) holds by (20)–(22), see Fig. 3. $\square$

Lemma 2.3. $\frac{d}{dh}\left(\frac{Q(h)}{P(h)}\right) > 0$ *for* $h > 0$ *and* $0 < \lambda \leqslant 1$.

Proof. By (7) it is equivalent to prove that $\frac{d}{dh}\left(\frac{I_2(h)}{I_1(h)}\right) > 0$ for $h > 0$ and $\lambda \in (0, 1]$. Let

$$\xi(x) = \frac{x^2 \Phi'(\tilde{x}) - \tilde{x}^2 \Phi'(x)}{x \Phi'(\tilde{x}) - \tilde{x} \Phi'(x)} = \frac{x\tilde{x} - 1}{x + \tilde{x} - 2\lambda},$$

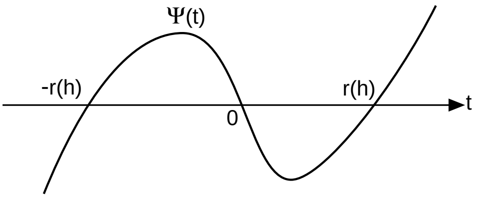

Fig. 3. The graph of $\Psi(t)$.

where $\tilde{x} = \tilde{x}(x)$ is defined as before. By Lemma 2.1, $x + \tilde{x} < \frac{4}{3}\lambda$, hence the function $\xi(x)$ is well defined for $x<0$. By using $\frac{d\tilde{x}}{dx} = \frac{\Phi'(x)}{\Phi'(\tilde{x})}$ we obtain

$$\xi'(x) = \frac{\eta(x)}{(x + \tilde{x} - 2\lambda)^2 \Phi'(\tilde{x})}, \tag{23}$$

where $\eta(x) = x^5 + \tilde{x}^5 - 4\lambda(x^4 + \tilde{x}^4) + (4\lambda^2 + 2)(x^3 + \tilde{x}^3) - 4\lambda(x^2 + \tilde{x}^2) + x + \tilde{x}$.

Using (10), (15) and

$$\begin{cases} x^5 + \tilde{x}^5 = u^5 - 5u^3 v + 5uv^2, \\ x^4 + \tilde{x}^4 = u^4 - 4u^2 v + 2v^2, \end{cases} \tag{24}$$

and changing v to the function of u in (12), we get

$$\eta(x) = -\frac{u(u^2 - 4\lambda u + 4)}{36(3u - 4\lambda)^2} [(3u - 4\lambda)^2 (9u^2 - 36\lambda u + 40\lambda^2) + 16\lambda(9 - 8\lambda^2)(5\lambda - 3u)],$$

which is negative for $0 < \lambda \leqslant 1$ and $0 < u < \frac{4\lambda}{3}$ (see (12)). On the other hand, $\Phi'(\tilde{x}) > 0$ since $\tilde{x} > 0$, hence $\xi'(x) < 0$ by (23), and this implies $\frac{d}{dh}\left(\frac{I_2(h)}{I_1(h)}\right) > 0$ by Corollary 2 of [LZ]. $\square$

Lemma 2.4. *$Q'(h) > 0$ for $h > 0$ and $\lambda \in [0, 1]$.*

Proof. We consider two cases separately.

Case 1: $\lambda \in (0, 1]$. From Lemma 2.2 and $P(0) = 0$ we have $P(h) > 0$ for $h > 0$. Obviously, $Q(h) > 0$ for $h > 0$. Hence the conclusion follows immediately from Lemmas 2.2 and 2.3.

Case 2: $\lambda = 0$. Γ_h is symmetric with respect to the y-axis. Hence for any $x<0$ we have $\tilde{x}(x) = -x > 0$, and

$$\bar{\xi}(x) = \frac{x^2 \Phi'(\tilde{x}) - \tilde{x}^2 \Phi'(x)}{\Phi'(\tilde{x}) - \Phi'(x)} = x^2.$$

The conclusion follows from $\bar{\xi}'(x) = 2x < 0$ and Corollary 2 of [LZ]. $\square$

Remark 2.5. As we mentioned at the beginning of Section 1, if system (2) has a degenerate global centre at $(0,0)$, then $b=c=0$, and a can be changed to 1. Γ_h is symmetric, hence $I_1(h)\equiv 0$. In the same way as in Case 2 of Lemma 2.4, we can prove $Q'(h)>0$ for $h>0$. Thus, the corresponding Abelian integral has at most one zero.

Lemma 2.6. *For $\lambda\in[0,1]$ and $h\to+\infty$ we have*

$$I_0(h)=2B\left(\frac{1}{4},\frac{3}{2}\right)h^{3/4}+o(h^{3/4}),$$

$$I_2(h)=4B\left(\frac{3}{4},\frac{3}{2}\right)h^{5/4}+o(h^{5/4}),$$

where $B(\alpha,\beta)$ is the Beta function.

Proof. Since $A(h)$ and $B(h)$ satisfy $H(x,0)=h$ (see (4) and Fig. 2), for $h\to+\infty$ we have

$$\begin{aligned}B(h)&=\sqrt{2}h^{1/4}\left(1-\frac{8\lambda}{3}\frac{1}{B(h)}+\frac{2}{B^2(h)}\right)^{-1/4}\\&=\sqrt{2}h^{1/4}\left(1+\frac{2\lambda}{3}\frac{1}{B(h)}+\frac{20\lambda^2-9}{18}\frac{1}{B^2(h)}\right.\\&\quad\left.+\frac{20\lambda^2-15}{9}\frac{1}{B^3(h)}+o\left(\frac{1}{B^3(h)}\right)\right),\end{aligned}\tag{25}$$

$$\begin{aligned}\frac{\sqrt{2}h^{1/4}}{B(h)}&=\left(1-\frac{8\lambda}{3}\frac{1}{B(h)}+\frac{2}{B^2(h)}\right)^{1/4}\\&=1-\frac{\sqrt{2}\lambda}{3}\frac{1}{h^{1/4}}+\frac{9-4\lambda^2}{36}\frac{1}{h^{1/2}}+o\left(\frac{1}{h^{1/2}}\right),\end{aligned}\tag{26}$$

$$\frac{\sqrt{2}h^{1/4}}{B^2(h)}=\frac{1}{\sqrt{2}h^{1/4}}-\frac{2\lambda}{3}\frac{1}{h^{1/2}}+o\left(\frac{1}{h^{1/2}}\right)\tag{27}$$

and

$$\frac{\sqrt{2}h^{1/4}}{B^3(h)}=\frac{1}{2h^{1/2}}+o\left(\frac{1}{h^{1/2}}\right).\tag{28}$$

Substituting (26)–(28) into (25) we obtain

$$B(h) = \sqrt{2}h^{1/4} + \frac{2\lambda}{3} + \frac{4\lambda^2 - 3}{6\sqrt{2}}\frac{1}{h^{1/4}} - \frac{(9-8\lambda^2)\lambda}{27}\frac{1}{h^{1/2}} + o\left(\frac{1}{h^{1/2}}\right), \tag{29}$$

Similarly, we have

$$A(h) = -\sqrt{2}h^{1/4} + \frac{2\lambda}{3} - \frac{4\lambda^2 - 3}{6\sqrt{2}}\frac{1}{h^{1/4}} - \frac{(9-8\lambda^2)\lambda}{27}\frac{1}{h^{1/2}} + o\left(\frac{1}{h^{1/2}}\right). \tag{30}$$

From the proof of Lemma 1(ii) in [DL2] it is easy to see that the computations of $I_0(h)$ and $I_2(h)$ only depend on the leading terms of $A(h)$ and $B(h)$ in h and the leading term of $H(x,0)$ in x, and they are independent of λ. Hence the asymptotic expressions of $I_0(h)$ and $I_2(h)$ in formula (13) of [DL2] are still true for $\lambda \neq 1$. $\square$

Remark 2.7. As an alternative proof, from (29) and (30) we get Lemma 2.1(ii).

Lemma 2.8. *For* $\lambda \in (0,1]$ *we have*

(i) $0 < P(h) < \frac{2\lambda}{3}$ *for* $h > 0$.

(ii) $P(h) - \frac{2\lambda}{3} = -\frac{\lambda(9-8\lambda^2)}{27}\left(1 + \frac{16}{15B(\frac{1}{4},\frac{3}{2})}\right)\frac{1}{\sqrt{h}} + o(\frac{1}{\sqrt{h}})$ *as* $h \to +\infty$.

(iii) $Q(h) = \frac{2B(\frac{3}{4},\frac{3}{2})}{B(\frac{1}{4},\frac{3}{2})}\sqrt{h} + o(\sqrt{h})$ *as* $h \to +\infty$.

Proof. By (7) and (16)

$$P(h) - s(h) = \frac{I_1(h) - s(h)I_0(h)}{I_0(h)} = \frac{2\int_{A(h)}^{B(h)}(x - s(h))y(x)\,dx}{I_0(h)}. \tag{31}$$

From (18), (19) and (31) we get $P(h) - s(h) < 0$ for $h > 0$.

Lemma 2.1(i) shows that $s(h) < \frac{2\lambda}{3}$ for $\lambda > 0$. Hence conclusion (i) is proved. Using (18) again we have

$$\int_{A(h)}^{B(h)}(x - s(h))y(x)\,dx = \int_0^{r(h)} t[y(s(h)+t) - y(s(h)-t)]\,dt, \tag{32}$$

where $y = y(x) \geqslant 0$ is defined by $H(x,y) = h$. Hence

$$\begin{aligned} &y(s(h)+t) - y(s(h)-t) \\ &\quad = [2(h - \Phi(s(h)+t))]^{1/2} - [2(h - \Phi(s(h)-t))]^{1/2} \end{aligned}$$

$$= \sqrt{2h}\left[\left(1-\frac{\Phi(s(h)+t)}{h}\right)^{1/2} - \left(1-\frac{\Phi(s(h)-t)}{h}\right)^{1/2}\right]$$
$$= \frac{1}{\sqrt{2h}}\Psi(t)\left(1+O\left(\frac{1}{\sqrt{h}}\right)\right) \quad \text{for } t\in[0,r(h)] \text{ and } h\to+\infty, \tag{33}$$

where $\Psi(t) = \Phi(s(h)-t) - \Phi(s(h)+t)$. Substituting (33) into (32) and using (21) we obtain

$$\int_{A(h)}^{B(h)} (x-s(h))y(x)\,dx = -\frac{8k}{15}(h^{3/4}+o(h^{3/4})) \quad \text{as } h\to+\infty, \tag{34}$$

where k is given in (22).

Note that

$$P(h) - \frac{2\lambda}{3} = (P(h)-s(h)) + \left(s(h)-\frac{2\lambda}{3}\right). \tag{35}$$

Using (31), (34), (22), (29), (30) and Lemma 2.6, from (35) we obtain conclusion (ii) of the lemma. Conclusion (iii) is a direct consequence of Lemma 2.6. □

From Lemmas 2.2–2.4 and 2.8 we have the following important property about the number of intersection points of the curve Σ_λ and the straight lines $\mathscr{L}_{\alpha,\beta}$, defined by (8) and (9), respectively.

Lemma 2.9. *For $\lambda\in(0,1]$ we have*

(i) *If $\alpha\leqslant 0$, then $\mathscr{L}_{\alpha,\beta}$ can cut $\Sigma_\lambda\backslash\{(0,0)\}$ at most once, and the intersection is transverse.*
(ii) *If the total number of intersection points of Σ_λ and $\mathscr{L}_{\alpha,\beta}$ is $k>1$, taking into account the multiplicity, then k must be even, and the corresponding Abelian integral $I(h)$ has $k+1$ zeros for $h\geqslant 0$, counting the multiplicity.*

Proof. Conclusion (i) follows from Lemma 2.3 which means that the polar angle of any point on Σ_λ is strictly increasing as h increases.

By conclusion (i), if $k>1$ then $\mathscr{L}_{\alpha,\beta}$ must cut the Q-axis below the origin. By Lemma 2.8, Σ_λ has an asymptotic line $\{P=\frac{2\lambda}{3}\}$ as $h\to+\infty$, hence Σ_λ must cut $\mathscr{L}_{\alpha,\beta}$ upwards at the most right intersection point. Therefore, k must be even (see Fig. 4). On the other hand, $h=0$ is always a zero of $I(h)$. Hence $I(h)$ has $k+1$ zeros, counting the multiplicity. Thus, conclusion (ii) is proved. □

To conclude this section, we give the differential equation satisfied by (h,P,Q), which can be obtained by taking $a=1$, $b=-2\lambda$ and $c=1$ in

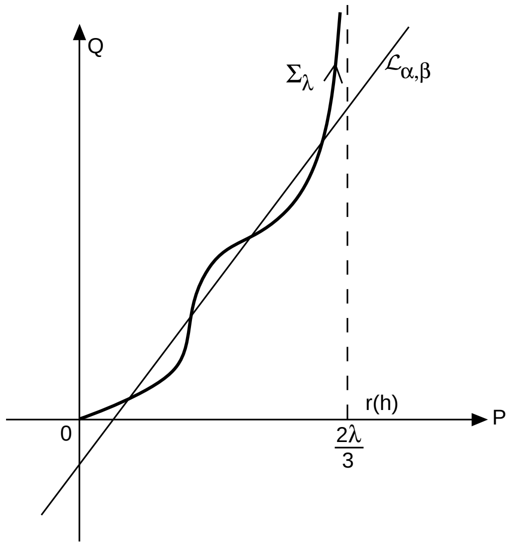

Fig. 4. The relative position of Σ_λ and $\mathscr{L}_{\alpha,\beta}$.

(11) of [DL1].

$$\begin{cases} \dot{h} = G(h), \\ \dot{P} = f(h,P,Q), \\ \dot{Q} = g(h,P,Q), \end{cases} \tag{36}$$

where

$$G(h) = 12h[144h^2 + 24(8\lambda^4 - 12\lambda^2 + 3)h - 8\lambda^2 + 9],$$

$$\begin{aligned} f(h,P,Q) = &-24(12h + 8\lambda^2 - 9)h + [432h^2 + 12(64\lambda^4 - 28\lambda^2 - 27)h \\ &+ 12(8\lambda^2 - 9)]P - 1440\lambda(\lambda^2 - 1)hQ - 4\lambda[12(4\lambda^2 + 3)h \\ &+ 7(8\lambda^2 - 9)]P^2 + [180(4\lambda^2 - 3)h + 15(8\lambda^2 - 9)]PQ, \end{aligned}$$

$$\begin{aligned} g(h,P,Q) = &-12[12(4\lambda^2 - 3)h + 8\lambda^2 - 9]h - 48\lambda(12h - 28\lambda^2 + 27)hP \\ &+ [864h^2 + 24(-80\lambda^4 + 92\lambda^2 - 9)h + 12(8\lambda^2 - 9)]Q \\ &- 4\lambda[12(4\lambda^2 + 3)h + 7(8\lambda^2 - 9)]PQ + [180(4\lambda^2 - 3)h \\ &+ 15(8\lambda^2 - 9)]Q^2. \end{aligned}$$

3. Study in the (h,ω)-plane

In order to study the number of triple points on Σ_λ, we also consider the second derivative of the Abelian integral (5)

$$I''(h) = \alpha I_0''(h) + \beta I_1''(h) + I_2''(h). \tag{37}$$

If $I_0''(h) \neq 0$, we define

$$\omega(h) = \frac{I_1''(h)}{I_0''(h)}, \qquad v(h) = \frac{I_2''(h)}{I_0''(h)}. \tag{38}$$

By taking $a = c = 1$ and $b = -2\lambda$ in (14) and (21) of [DL1], we get the equation of (h,ω):

$$\begin{cases} \dot{h} = G(h), \\ \dot{\omega} = \varphi(h,\omega), \end{cases} \tag{39}$$

where $G(h)$ is the same as in (36), and

$$\begin{aligned}
&\varphi(h,\omega)\\
&\quad = \left[\frac{1944(4\lambda^2-3)}{\lambda(8\lambda^2-9)}h^2 - \frac{12(16\lambda^4-18\lambda^2-27)}{\lambda}h + \frac{(40\lambda^2-9)(8\lambda^2-9)}{2\lambda}\right]\omega^2\\
&\qquad + \left[\frac{1728(9-10\lambda^2)}{8\lambda^2-9}h^2 + 96(8\lambda^2-11)\lambda^2 h - 12(8\lambda^2-9)\right]\omega\\
&\qquad + \frac{288\lambda(28\lambda^2-27)}{8\lambda^2-9}h^2 - 24\lambda(8\lambda^2-9)h,
\end{aligned}$$

and the expressions of ω and v as functions of P, Q and h:

$$\begin{cases} \omega(h) = \dfrac{2\lambda[108h + (9-8\lambda^2)(28\lambda P(h) - 15Q(h) - 12)]}{324h + (9-8\lambda^2)(24\lambda P(h) - 40\lambda^2 + 9)}, \\[2ex] v(h) = \dfrac{12h[60\lambda P(h) - 45Q(h) + 4(10\lambda^2-9)] + (9-8\lambda^2)(28\lambda P(h) - 15Q(h) - 12)}{324h + (9-8\lambda^2)(24\lambda P(h) - 40\lambda^2 + 9)}. \end{cases} \tag{40}$$

From $P(0) = Q(0) = 0$, (40) and Lemma 2.8 we immediately obtain

Lemma 3.1.

(i) $\omega(0) = \frac{24\lambda}{40\lambda^2-9}$, $v(0) = \frac{12}{40\lambda^2-9}$.

(ii) *$\omega(h) \to \frac{2\lambda}{3}$ and $v(h) \to -\infty$ as $h \to +\infty$.*

(iii) *$\omega(h)$ is strictly increasing and less than $\frac{2\lambda}{3}$ for $h \gg 1$.* $\square$

Taking $a = c = 1$ and $b = -2\lambda$ in (9) of [DL1] we have for $\lambda \in (0, 1]$

$$I_2''(h) = \frac{12}{8\lambda^2 - 9} hI_0''(h) + \frac{1}{2\lambda}\left(\frac{36}{9 - 8\lambda^2}h + 1\right)I_1''(h). \tag{41}$$

Hence, if $I_0''(h) \neq 0$, then from (37), (38) and (41) we have

$$I''(h) = \left(\alpha + \frac{12h}{8\lambda^2 - 9}\right)I_0''(h) + \left(\beta + \frac{1}{2\lambda}\left(\frac{36h}{9 - 8\lambda^2} + 1\right)\right)I_1''(h). \tag{42}$$

Lemma 3.2. *For any $h > 0$ and $\lambda > 0$, if $I_0''(h) = 0$ then $I_1''(h) \neq 0$.*

Proof. From the first equation of (7) in [DL1] we have

$$3I_0(h) = 4hI_0'(h) - \frac{2\lambda}{3}I_1'(h) + \frac{4\lambda^2 - 3}{3}I_2'(h).$$

Making a derivative with respect to h, we get

$$I_0'(h) = -4hI_0''(h) + \frac{2\lambda}{3}I_1''(h) - \frac{4\lambda^2 - 3}{3}I_2''(h).$$

If for some $h > 0$ and $\lambda > 0$ $I_0''(h) = I_1''(h) = 0$, then from (41) we have $I_2''(h) = 0$, hence we obtain $I_0'(h) = 0$ which is impossible, since $I_0'(h)$ is the period of Γ_h. $\square$

We note that if $\lambda = 1$ then system $(3)_0$ has a cuspidal loop corresponding to $h = \frac{1}{12}$. Since $I_0'(h) = \int_{\Gamma_h} \frac{1}{y} dx$ is the period function of Γ_h, we must have $I_0'(h) \to +\infty$ as $h \to \frac{1}{12} \pm 0$. In this case the period function has no critical point (see [CS,G]), hence, $I_0''(h)(h - \frac{1}{12}) < 0$ for $h \neq \frac{1}{12}$. If $\lambda \in (0, 1)$ and $|\lambda - 1| \ll 1$, then $I_0(h) \in C^\infty[0, +\infty)$. By continuity $I_0'(h)$ has a maximum at some value h_λ^* near $\frac{1}{12}$ which is a zero point of $I_0''(h)$. By [G], h_λ^* is unique, and $I_0''(h)(h - h_\lambda^*) < 0$ for $h \neq h_\lambda^*$.

It is easy to see that for $\lambda \neq \frac{3}{2\sqrt{10}}$ system (39) has two singularities: a saddle point at $A(0, \frac{24\lambda}{40\lambda^2 - 9})$ and a node at $(0, 0)$. By Lemma 3.1 the orbit $C_\omega : \omega = \omega(h)$ of system (39), which we look for, comes from the saddle point A as its unstable manifold, and goes to the asymptotic line $\{\omega = \frac{2\lambda}{3}\}$, monotonically increasing for $h \gg 1$. More precisely, we have the following.

Lemma 3.3.

(i) *If $\frac{3}{2\sqrt{10}} < \lambda < 1$, then there exists a unique $h_\lambda^* > 0$, such that $I_0''(h_\lambda^*) = 0$ and $I_0''(h)(h - h_\lambda^*) < 0$ for $h \neq h_\lambda^*$. In this case $C_\omega = C_\omega^1 \cup C_\omega^2$, where C_ω^1 is the unstable manifold of the saddle point A, tending to $+\infty$ as $h \to h_\lambda^* - 0$; C_ω^2 tends to $-\infty$ as $h \to h_\lambda^* + 0$, and tends to its asymptotic line $\{\omega = \frac{2\lambda}{3}\}$, strictly increasing for $h \gg 1$.*

(ii) *If* $0<\lambda<\frac{3}{2\sqrt{10}}$, *then* C_ω *is globally defined for* $h>0$. *It comes from the saddle point* A, *and tends to the line* $\{\omega=\frac{2\lambda}{3}\}$, *strictly increasing for* $h\gg 1$.

(iii) *If* $\lambda=\frac{3}{2\sqrt{10}}$, *then the behaviour of* C_ω *is similar as in case* (ii), *the only difference is that* $\omega(h)\to-\infty$ *as* $h\to 0+0$.

Proof. The conclusions follows from the following facts:

(1) By Lemma 3.1 C_ω should start from the saddle point A (if $\lambda\neq\frac{3}{2\sqrt{10}}$), and tends to the line $\{\omega=\frac{2\lambda}{3}\}$, strictly increasing for $h\gg 1$.

(2) By Lemma 3.2, C_ω is globally defined for $h>0$ (respectively, C_ω is discontinuous at some point) if and only if $I_0''(h)$ has no zero point (resp., has a zero point). By Chow and Sanders [CS] and Gavrilov [G], the zero point is unique if it exists.

(3) The study of the singular point at infinity shows that the unstable manifold of the saddle point A goes to the singular point at infinity D_1 in positive ω-direction and forms an elliptic sector if $\frac{3}{2\sqrt{10}}<\lambda<1$, shown in Fig. 5(a); it goes to the singular point at infinity D_2 in positive h-direction if $0<\lambda<\frac{3}{2\sqrt{10}}$, shown in Fig. 5(c). The case $\lambda=\frac{3}{2\sqrt{10}}$ is the critical case between these two cases, shown in Fig. 5(b). $\square$

Note that $G(h)$ is strictly positive for $h>0$; and $\varphi(h,\omega)$ is a polynomial of degree two both in h and ω, hence system (39) has the following property:

Property (P). For any constant c, the straight line $\{h=c\}$ or $\{\omega=c\}$ cuts the 0-cline of system (39) at most twice.

Lemma 3.4. *For different* λ, C_ω *is as shown in* Fig. 6, *where* A *is the saddle point of system* (39). *More precisely, if* $\lambda\in(0,\frac{3}{2\sqrt{10}}]$, *then* C_ω *is globally defined, it stays below the line* $\{\omega=\frac{2\lambda}{3}\}$ *and tends to this line as* $h\to+\infty$; *if* $\lambda\in(\frac{3}{2\sqrt{10}},1)$ *then* $C_\omega=C_\omega^1\cup C_\omega^2$, *where* C_ω^1 *is strictly increasing,* C_ω^2 *tends to the line* $\{\omega=\frac{2\lambda}{3}\}$ *as* $h\to+\infty$, *strictly monotonically increasing for* $h\gg 1$. *Besides, when* $\lambda\in(\frac{3}{2\sqrt{10}},\frac{\sqrt{3}}{2}]$, C_ω^2 *stays below the line* $\{\omega=\frac{2\lambda}{3}\}$; *when* $\lambda\in(\frac{\sqrt{3}}{2},1)$, C_ω^2 *may cut this line transversally at exactly two points, or it is tangent to this line at one point.*

Proof. The basic behaviour of C_ω follows from Lemma 3.3. If $\lambda\in(0,\frac{3}{2\sqrt{10}}]$, C_ω is globally defined for $h>0$. The proof of the fact that C_ω stays below the line $\{\omega=\frac{2\lambda}{3}\}$ is the same as below, where we will prove this property for C_ω^2 when $\lambda\leqslant\frac{\sqrt{3}}{2}$.

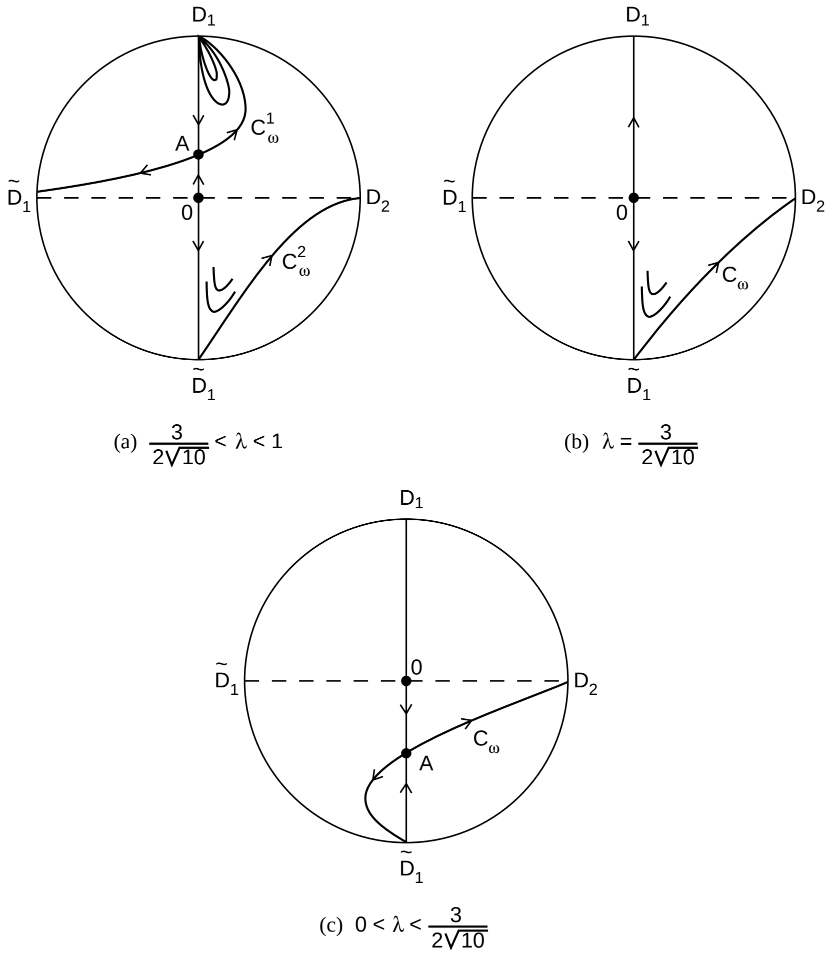

Fig. 5. Singular points at infinity.

If $\lambda \in (\frac{3}{2\sqrt{10}}, 1)$, then a branch C_0 of the 0-cline of system (39), passing through the saddle point A, has an asymptotic line $\{h = h_1\}$, where

$$h_1 = \frac{2(9 - 8\lambda^2)(40\lambda^2 - 9)}{648 + 432\lambda^2 - 384\lambda^4 + 48\lambda\sqrt{64\lambda^6 - 144\lambda^4 - 1215\lambda^2 + 1296}} > 0.$$

C_0 must be strictly increasing. Otherwise, it contradicts property (P), see Fig. 7(b). On the other hand, calculation shows that the slope of C_ω^1 at point A is positive, and

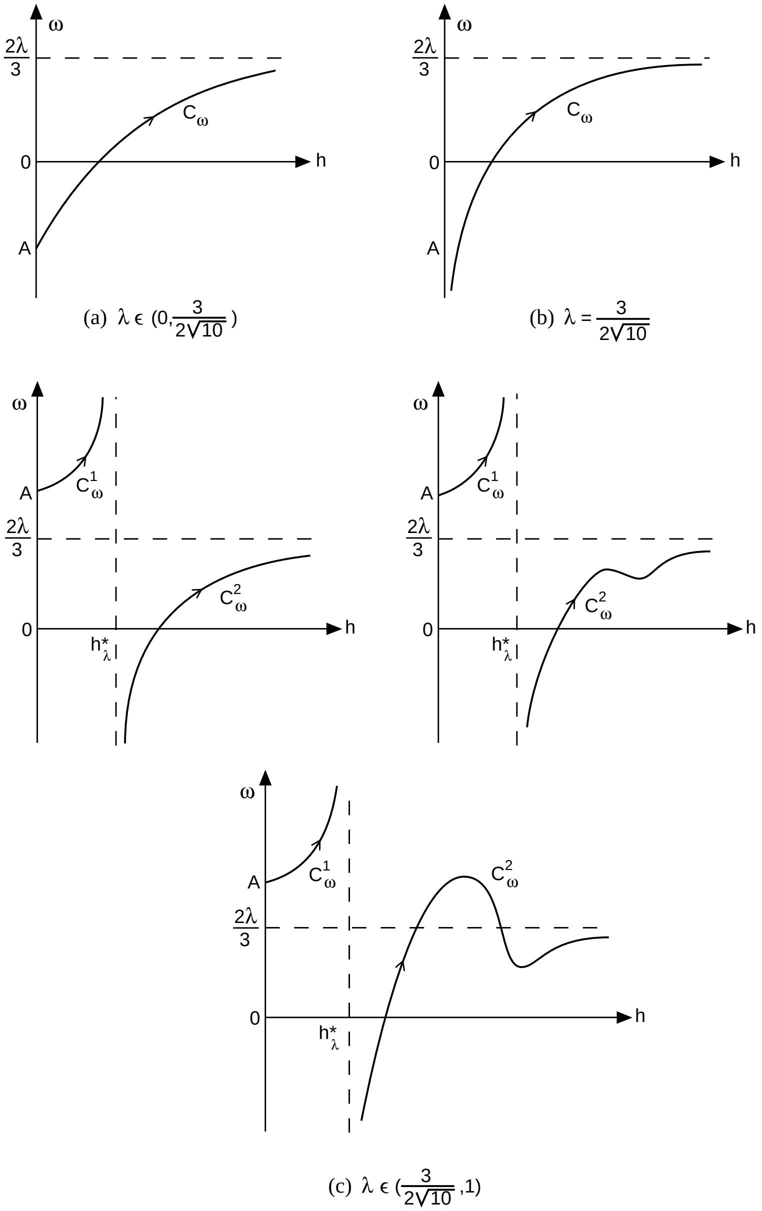

Fig. 6. The behaviour of C_ω.

it is equal to half of the slope of C_0 at A. These two facts imply that C_ω^1 must stay below C_0, shown in Fig. 7(a), and it is strictly increasing. Otherwise, it contradicts the property (P), see Fig. 7(c).

It is easy to see that there is a unique point M on the line $\{\omega = \frac{2\lambda}{3}\}$, corresponding to $h = \frac{9-8\lambda^2}{12(4\lambda^2-3)}$ (if $\lambda \neq \frac{\sqrt{3}}{2}$), at which the vector field (39) is tangent to the line $\{\omega = \frac{2\lambda}{3}\}$.

If $\lambda \leqslant \frac{\sqrt{3}}{2}$, then the vector field (39) is pointing upwards at any point of this line for $h > 0$. If C_ω^2 cuts this line, then it can never come back below it, contradicting Lemma 3.1(iii). If $\lambda > \frac{\sqrt{3}}{2}$, then C_ω^2 may cut this line transversally at two points, or C_ω^2 may be tangent to this line at the point M. $\square$

Remark 3.5. It is possible to prove that C_ω (for $\lambda \in (0, \frac{3}{2\sqrt{10}}]$) and C_ω^2 (for $\lambda \in (\frac{3}{2\sqrt{10}}, \frac{\sqrt{3}}{2}]$) are strictly increasing, while C_ω^2 (for $\lambda \in (\frac{\sqrt{3}}{2}, 1)$) is either strictly increasing, or has one maximum and one minimum. We will not carry it out, since we do not need it in the further study.

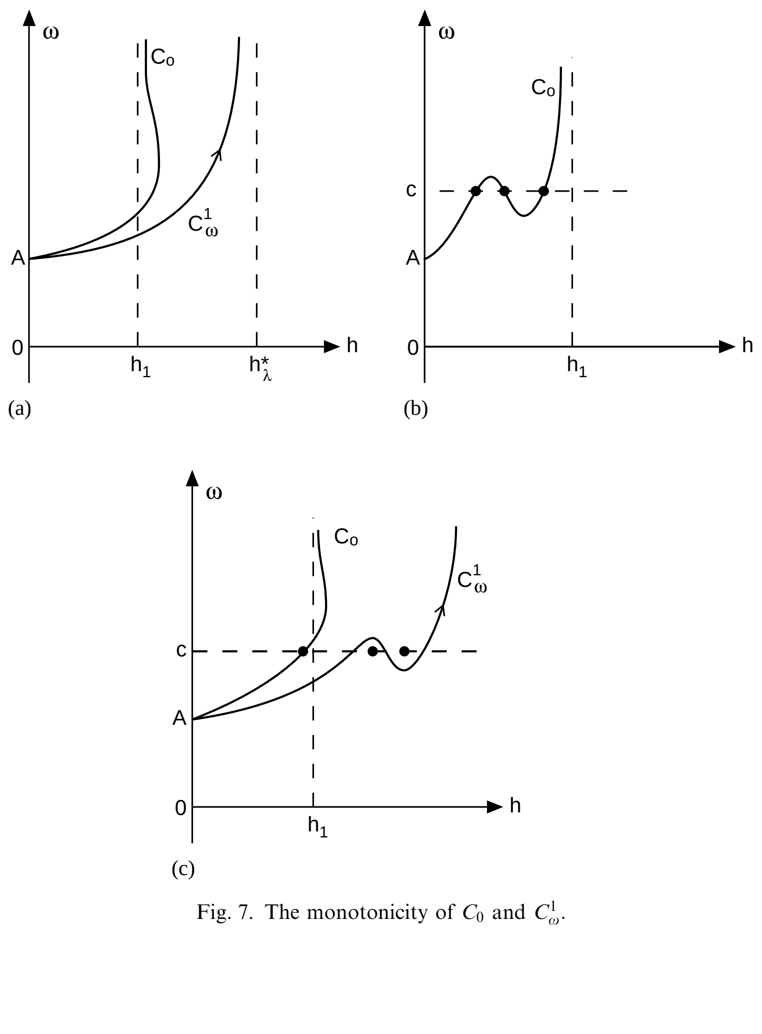

Fig. 7. The monotonicity of C_0 and C_ω^1.

Lemma 3.6. *$I''(h^*_\lambda) = 0$ if and only if*

$$\beta = -\frac{1}{2\lambda}\left(\frac{36}{9-8\lambda^2}h^*_\lambda + 1\right). \tag{43}$$

Proof. The conclusion follows from Lemma 3.3(i), Lemma 3.2 and (42). □

Taking $a = c = 1$ and $b = -2\lambda$ in (13) of [DL1] we have

$$G(h)\frac{d}{dh}\begin{pmatrix} I_0'' \\ I_1'' \end{pmatrix} = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}\begin{pmatrix} I_0'' \\ I_1'' \end{pmatrix}, \tag{44}$$

where $G(h)$ is the same as in (39), and

$$b_{00} = \frac{1728(7\lambda^2-9)}{9-8\lambda^2}h^2 - 48(56\lambda^4 - 83\lambda^2 + 18)h + 12(8\lambda^2-9),$$

$$b_{01} = \frac{1944(4\lambda^2-3)}{\lambda(9-8\lambda^2)}h^2 + \frac{12(16\lambda^4 - 18\lambda^2 - 27)}{\lambda}h - \frac{(8\lambda^2-9)(40\lambda^2-9)}{2\lambda},$$

$$b_{10} = \frac{288\lambda(28\lambda^2-27)}{8\lambda^2-9}h^2 - 24\lambda(8\lambda^2-9)h,$$

$$b_{11} = -\frac{1728(17\lambda^2-18)}{8\lambda^2-9}h^2 - 48(8\lambda^2-9)(5\lambda^2-2)h.$$

Lemma 3.7. *$I''(h^*_\lambda) = I'''(h^*_\lambda) = 0$ if and only if* (43) *holds and*

$$\alpha = \left.\frac{6(2\lambda h b_{01} - 3G(h))}{\lambda(9-8\lambda^2)b_{01}}\right|_{h=h^*_\lambda}. \tag{45}$$

Proof. The conclusion follows from (42), (44), Lemmas 3.2 and 3.6. □

Remark 3.8. A straightforward calculation shows that if $I''(h^*_\lambda) = I'''(h^*_\lambda) = I^{(4)}(h^*_\lambda) = 0$, then (43) and (45) hold, and

$$4\lambda b_{01}^2 - 3((G'(h) + b_{11} - b_{00})b_{01} - G(h)b_{01}'(h))|_{h=h^*_\lambda} = 0. \tag{46}$$

We will prove in Lemma 4.7 that if (43) and (45) hold then (46) is impossible, hence the tangency of Σ_λ at the point $(P(h^*_\lambda), Q(h^*_\lambda))$ is at most quadruple.

4. Proof of conclusion (B1)

We will prove conclusion (B1) of Theorem B by using different methods for different ranges of λ, α and β.

4.1. *The case* $\lambda \in (\frac{3}{2\sqrt{10}}, 1)$ *and* $18\alpha + 12\lambda\beta + 20\lambda^2 - 9 \geqslant 0$

We suppose the contrary: there are constants α, β and $\lambda \in (0, 1)$, such that the curve Σ_λ and the straight line $\mathscr{L}_{\alpha,\beta}$ have more than 4 intersection points, counting the multiplicity. In Section 6, we will prove that $\lim_{h\to 0+} \frac{Q'(h)}{P'(h)} = \frac{1}{2\lambda}$. By using this fact as well as Lemmas 2.2–2.4 and 2.9, we easily obtain that α and β must satisfy

$$\alpha > 0, \quad \beta < -\frac{1}{2\lambda}, \tag{47}$$

and the total number of intersection points of Σ_λ and $\mathscr{L}_{\alpha,\beta}$ is at least 6, see Fig. 8(a). This is equivalent to say that in (h, P, Q)-space the trajectory of system (36)

$$\widetilde{\Sigma}_\lambda = \{(h, P, Q) \mid P = P(h), Q = Q(h), h > 0\}$$

and the plane

$$\widetilde{\mathscr{L}}_{\alpha,\beta} = \{(h, P, Q) \mid (P, Q) \in \mathscr{L}_{\alpha,\beta}, h > 0\}$$

have at least 6 intersection points, as shown in Fig. 8(b).

Hence there exist at least 5 points $\widetilde{M}_i \in \widetilde{C}_P = \{(h, P, Q) \in \widetilde{\mathscr{L}}_{\alpha,\beta} \mid P = P(h)\}$, at which the vector field (36) is tangent to $\widetilde{\mathscr{L}}_{\alpha,\beta}$.

It is easy to see that $\{\widetilde{M}_i\}$ satisfy

$$\begin{cases} \alpha + \beta P + Q = 0, \\ \beta \dot{P} + \dot{Q} = 0. \end{cases} \tag{48}$$

By using (36) and eliminating Q from (48), we obtain

$$(\xi_2 h^2 + \xi_1 h + \xi_0) - (\eta_2 h^2 + \eta_1 h + \eta_0)P = 0.$$

If the two polynomials of h have a double common root, then it is obvious that (48) has at most 3 solutions in h since $P'(h) > 0$. If they have a simple common root, then the discussion is the same as below, and the situation is simpler. So we suppose that the two polynomials have no common root, hence

$$P = W(h) = \frac{\xi_2 h^2 + \xi_1 h + \xi_0}{\eta_2 h^2 + \eta_1 h + \eta_0}, \tag{49}$$

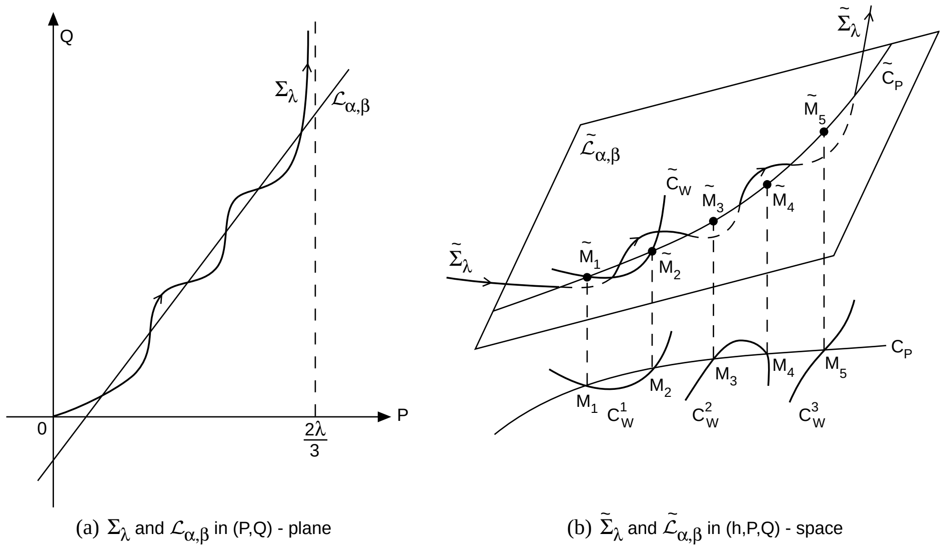

Fig. 8. Situation where Σ_λ and $\mathscr{L}_{\alpha,\beta}$ have more than 4 intersection points.

where $\{\xi_i\}$ and $\{\eta_i\}$ depend on λ, α and β, and

$$\begin{cases} \xi_2 &= 144(6\alpha + 2\lambda\beta + 4\lambda^2 - 3), \\ \eta_2 &= -144(3\beta + 4\lambda), \\ \eta_1 &= 48\lambda(4\lambda^2 + 3)\alpha + 12(224\lambda^4 - 212\lambda^2 - 9)\beta \\ & \quad +180(4\lambda^2 - 3)\alpha\beta + 1440\lambda(\lambda^2 - 1)\beta^2 + 48\lambda(28\lambda^2 - 27), \\ \eta_0 &= (8\lambda^2 - 9)(15\beta + 28\lambda)\alpha. \end{cases} \tag{50}$$

Thus, $\widetilde{M}_i \in \widetilde{C}_P \cap \widetilde{C}_W$, where $\widetilde{C}_W = \{(h, P, Q) \in \widetilde{\mathscr{L}}_{\alpha,\beta} \mid P = W(h)\}$.

In other words, at each $\widetilde{M}_i$ the vector field (36) is tangent to $\widetilde{C}_P$. In order to study the number of $\{\widetilde{M}_i\}$, we will also consider some points $\widetilde{N}_i \in \widetilde{C}_W$, at which the vector field (36) is tangent to $\widetilde{C}_W$ ($\widetilde{N}_i$ may coincide with $\widetilde{M}_i$ or $\widetilde{M}_{i+1}$). Such tangent points are given by the zeros of the following function:

$$\dot{P} - W'(h)\dot{h}|_{\substack{Q=-\alpha-\beta P \\ P=W(h)}} = \frac{G(h)(\zeta_3 h^3 + \zeta_2 h^2 + \zeta_1 h + \zeta_0)}{(\eta_2 h^2 + \eta_1 h + \eta_0)^2}, \tag{51}$$

where $G(h) > 0$ for $h > 0$ (see (36)), $\{\zeta_i\}$ depend on λ, α and β, and

$$\begin{cases} \zeta_3 = -1728(3\beta + 4\lambda)(18\alpha + 12\lambda\beta + 20\lambda^2 - 9), \\ \zeta_0 = -5\alpha^2(8\lambda^2 - 9)^2(72\lambda\alpha + 3(40\lambda^2 - 9)\beta + 4\lambda(56\lambda^2 - 27)). \end{cases} \tag{52}$$

Making a projection of the curves $\widetilde{C}_P$ and $\widetilde{C}_W$ onto the (h,P)-plane, we obtain, respectively, the curves $C_P = \{(h,P) \mid P = P(h)\}$ and $C_W = \{(h,P) \mid P = W(h)\}$. $M_i \in C_P \cap C_W$ is just the projection of $\widetilde{M}_i$ $(i = 1, \dots, 5)$, and the correspondence is obviously one to one. For simplicity, we will say "the tangent point on C_W" (with respect to the vector field (36)), which makes sense on $\widetilde{\mathscr{L}}_{\alpha,\beta}$ as we explained above. We will prove that C_P and C_W have at most 4 intersection points, counting the multiplicity. This contradiction gives a proof of conclusion (B1) of Theorem B in the case under consideration.

We know from Lemmas 2.2 and 2.8 that the curve C_P is strictly increasing for $h > 0$, and has an asymptotic line $\{P = \frac{2\lambda}{3}\}$ as $h \to +\infty$. From (49) we see that the curve C_W consists of at most 3 branches $C_W^{(i)}$, $i = 1, 2, 3$.

If $\eta_2 \neq 0$, then $W(h) \to \frac{\xi_2}{\eta_2}$ as $h \to \pm\infty$, and

$$\tau = \frac{\xi_2}{\eta_2} - \frac{2\lambda}{3} = -\frac{18\alpha + 12\lambda\beta + 20\lambda^2 - 9}{3(3\beta + 4\lambda)}. \tag{53}$$

The following Lemma is obviously true by (49), but it is very important for our further study.

Lemma 4.1. *For any constant c, the straight line $\{P = c\}$ in (h,P)-plane cuts the curve C_W at most twice.*

Lemma 4.2. *If C_W has 3 branches and the curve C_P meets all of them, then $\tau < 0$ and all $C_W^{(i)}$ $(i = 1, 2, 3)$ must be strictly decreasing. Hence C_P and C_W have exact 3 intersection points, counting the multiplicity.*

Proof. Since C_W has 3 branches, we have $\eta_2 \neq 0$, and the numerator and denominator of $W(h)$ have no common factor. By Lemma 4.1, each branch $C_W^{(i)}$ either is strictly monotone, or has unique extreme point.

Let us first look at the middle branch $C_W^{(2)}$. If $C_W^{(2)}$ meets C_P and has an extreme point, then either $C_W^{(1)}$ or $C_W^{(3)}$ does not meet C_P, see Fig. 9(a) and (b). Here we also need to use the Lemma 4.1 and the monotonicity of C_P.

If $C_W^{(2)}$ is strictly increasing, then $C_W^{(1)}$ and $C_W^{(3)}$ must be also strictly increasing (by Lemma 4.1) and either $C_W^{(3)}$ or $C_W^{(1)}$ does not meet C_P, depending on $\frac{\xi_2}{\eta_2} \leqslant P(h_1)$ or $\frac{\xi_2}{\eta_2} > P(h_1)$, where h_1 is the smaller positive root of $\eta_2 h^2 + \eta_1 h + \eta_0 = 0$, see Fig. 9(c) and (d). Hence, the only possibility of C_P meeting all 3 branches $C_W^{(i)}$ is that all $C_W^{(i)}$ $(i = 1, 2, 3)$ are strictly decreasing, and $0 < \frac{\xi_2}{\eta_2} < \frac{2\lambda}{3}$, see Fig. 9(e). $\square$

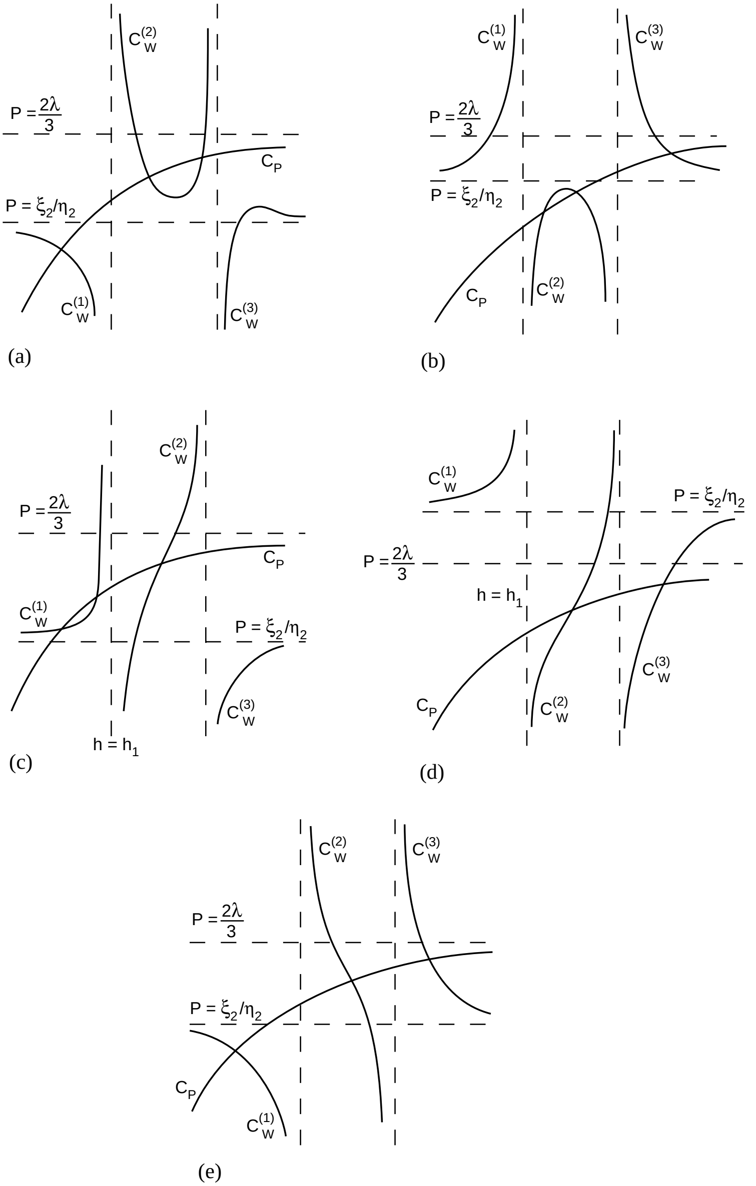

Fig. 9. Relative position of C_P and C_W in case C_W has 3 branches.

We denote by $\#(C_P \cap C_W)$ the number of intersection points of C_P and C_W, taking into account the multiplicity.

Lemma 4.3. *If the numerator of* (51) *has at most two positive zeros*, *then* $\#(C_P \cap C_W) \leqslant 4$.

Proof. If C_P meets C_W only at one branch, and $\#(C_P \cap C_W) \geqslant 5$, then along this branch of C_W there exist at least 4 "tangent points", contradicting the condition of the Lemma.

Suppose that C_P meets C_W at exactly 2 branches, for example $C_W^{(1)}$ and $C_W^{(2)}$, $\#(C_P \cap C_W^{(1)}) = n_1$, $\#(C_P \cap C_W^{(2)}) = n_2$, and $n_1 + n_2 \geqslant 5$. If n_1 or n_2 is equal to 1, then the other is at least 4; if both are $\geqslant 2$, then one of the two is at least 3. This implies that the "tangent points" on C_W is at least 3, also leading to a contradiction.

If C_P meets C_W at 3 branches, then the conclusion follows from Lemma 4.2. $\square$

Lemma 4.4. *If* $\zeta_3 \geqslant 0$, *then* $\#(C_P \cap C_W) \leqslant 4$.

Proof. If $\zeta_3 = 0$, then by (51) and Lemma 4.3 the conclusion is obviously true.

Suppose $\zeta_3 > 0$, then by (52) and (53) $\tau = \frac{\xi_2}{\eta_2} - \frac{2\lambda}{3} > 0$. If C_P meets the most right branch of C_W, and the most right intersection point corresponds to $h = h_r$, then there is at least one $h' \in [h_r, +\infty)$, such that $h = h'$ is a positive root of the numerator of (51), since both ζ_3 and τ are positive (see Fig. 10(a)). In Fig. 10 and in the rest of the proof we denote the most right branch of C_W by $C_W^{(3)}$ even in case C_W has less than 3 branches.

Hence, by the same arguments as in the proof of Lemma 4.3, we get $\#(C_P \cap C_W) \leqslant 4$. If $C_P \cap C_W^{(3)} = \phi$, then by the first part of the proof of Lemma 4.3, we only need to consider the case that C_W has 3 branches and that both $C_P \cap C_W^{(1)} \neq \phi$ and $C_P \cap C_W^{(2)} \neq \phi$. By Lemma 4.1 and $\tau > 0$, this implies that both $C_W^{(1)}$ and $C_W^{(2)}$ are strictly decreasing, hence $\#(C_P \cap C_W) = 2$, see Fig. 10(b). $\square$

Let $\zeta_3' = 18\alpha + 12\lambda\beta + 20\lambda^2 - 9$, then by (50) and (52) we have

$$\zeta_3 = 12\eta_2\zeta_3'. \tag{54}$$

Lemma 4.5. *If* $\lambda \in (\frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2})$ *and* (α, β) *belongs to the region*

$$D: \eta_2 < 0, \ \zeta_3' > 0 \quad \textit{and} \quad \zeta_0 > 0, \tag{55}$$

then $\#(C_P \cap C_W) \leqslant 4$.

Proof. The region D is bounded by the three straight lines $\{\eta_2 = 0\}$, $\{\zeta_3' = 0\}$ and $\{\zeta_0 = 0\}$, shown in Fig. 11.

By (50) if $(\alpha, \beta) \in D$, then $\eta_2 < 0$ and $\eta_0 < 0$. Let us show that $(\alpha, \beta) \in D$ also implies $\eta_1 < 0$, hence C_P meets at most one branch of C_W, and the conclusion follows from (51), which means there are at most 3 "tangent points" on C_W.

From (50) it is easy to see that $\eta_1 = 0$ defines a hyperbola and any straight line $\{\beta = c\}$ cuts only one branch of it. Eliminating β from $\eta_1 = 0$ and $\zeta_0 = 0$, we obtain

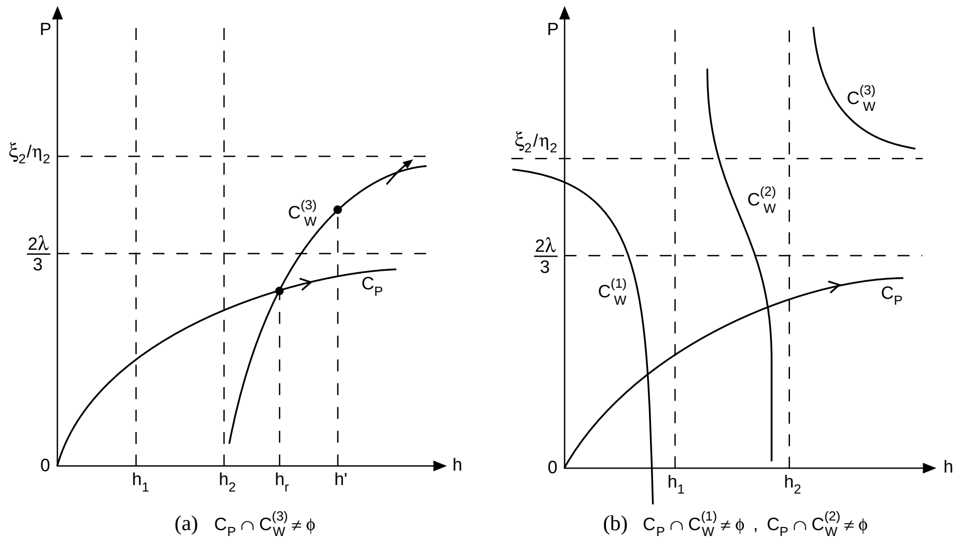

Fig. 10. Relative position of C_P and C_W in case $\zeta_3 > 0$.

a quadratic form of α, which has no root in $\alpha \in (0, +\infty)$ for $\lambda \in (\frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2})$, this e.g. follows from the Fourier–Budan rule. Denote by M and N the intersection points of $\{\zeta_0 = 0\}$ and $\{\eta_1 = 0\}$ with $\{\eta_2 = 0\}$, respectively, then $\alpha_M = \frac{1}{9}(9 - 8\lambda^2)$ and $\alpha_N = \frac{4}{3}(1 - \lambda^2)$. Since $\alpha_N - \alpha_M = \frac{1}{9}(3 - 4\lambda^2) > 0$ for $\lambda < \frac{\sqrt{3}}{2}$, $\{\eta_1 = 0\}$ is entirely located right to $\{\zeta_0 = 0\}$, see Fig. 11. It is easy to check now that for any $(\alpha, \beta) \in D$, we must have $\eta_1 < 0$. $\square$

Proposition 4.6. *If* $\lambda \in (\frac{3}{2\sqrt{10}}, 1)$ *and* $\zeta_3' \geqslant 0$, *then* $\#(C_P \cap C_W) \leqslant 4$.

Proof. If $\eta_2 = 0$, then by (50) and (52) $\zeta_3 = 0$, and the conclusion follows from Lemma 4.3. We suppose $\eta_2 \neq 0$. For different ranges of λ, the relative positions of the lines $\{\eta_2 = 0\}$, $\{\zeta_3' = 0\}$, $\{\zeta_0 = 0\}$ and $\{\beta = -\frac{1}{2\lambda}\}$ are shown in Fig. 12(a)–(e).

If (α, β) belongs to the shaded region in Fig. 12, then $\zeta_3\zeta_0 \geqslant 0$ (the equality corresponds to $\zeta_3' = 0$ or $\zeta_0 = 0$). Hence (51) has at most two positive roots, and the proposition follows by Lemma 4.3.

If $(\alpha, \beta) \in$ region E, then $\zeta_3 > 0$ by (54), and the proposition follows by Lemma 4.4.

If $(\alpha, \beta) \in$ region D (it exists only in Fig. 12(c)), then the conclusion follows by Lemma 4.5.

We remark here that we only need to consider (α, β) in domain (47); in case (b) the intersection point of $\{\zeta_3' = 0\}$ and $\{\zeta_0 = 0\}$ is located on the line $\{\eta_2 = 0\}$, and in case (d) the two lines $\{\zeta_3' = 0\}$ and $\{\zeta_0 = 0\}$ are parallel. $\square$

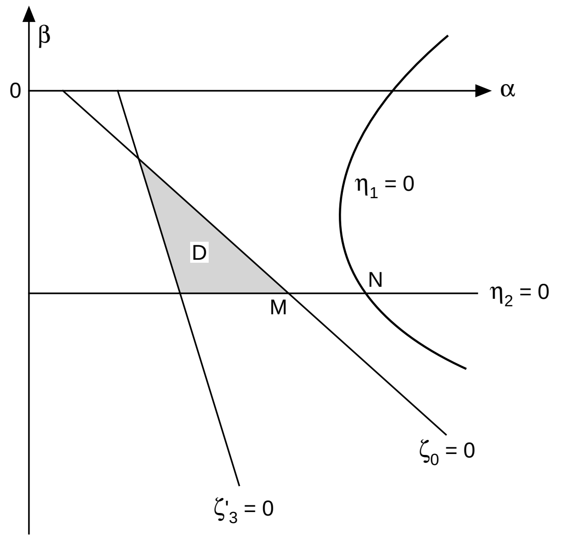

Fig. 11. The region D.

4.2. *The case* $\lambda \in \left(0, \frac{3}{2\sqrt{10}}\right]$, *or* $\lambda \in \left(\frac{3}{2\sqrt{10}}, 1\right)$ *and* $18\alpha + 12\lambda\beta + 20\lambda^2 - 9 < 0$

We suppose that conclusion (B1) of Theorem B is not true, then there are constants α, β and $\lambda \in (0,1)$ such that the straight line $\mathscr{L}_{\alpha,\beta}$ and the curve Σ_λ have more than 4 intersection points, including multiplicity. Then (α, β) satisfies (47), and by Lemma 2.9 $I(h)$ has at least 7 zeros for $h \geqslant 0$. Hence $I''(h)$ has at least 5 zeros for $h > 0$.

If $h \neq h_\lambda^*$, then $I_0''(h) \neq 0$. By (37), (38) and (42)

$$I''(h) = I_0''(h)\left[\alpha + \frac{12h}{8\lambda^2 - 9} + \left(\beta + \frac{1}{2\lambda}\left(\frac{36h}{9 - 8\lambda^2} + 1\right)\right)\omega(h)\right]. \tag{56}$$

Let

$$\bar{h}_\lambda = -\frac{9 - 8\lambda^2}{36}(2\lambda\beta + 1). \tag{57}$$

If $3\alpha + 2\lambda\beta + 1 = 0$, then (56) becomes

$$I''(h) = \frac{18 I_0''(h)}{\lambda(9 - 8\lambda^2)}(h - \bar{h}_\lambda)\left(\omega(h) - \frac{2\lambda}{3}\right), \tag{58}$$

and from (42) and Lemma 3.2 it is clear that h_λ^* is not a zero of $I''(h)$ unless $h_\lambda^* = \bar{h}_\lambda$, in which case h_λ^* is at most a double zero of $I''(h)$ (see Lemma 4.7). By (58) and

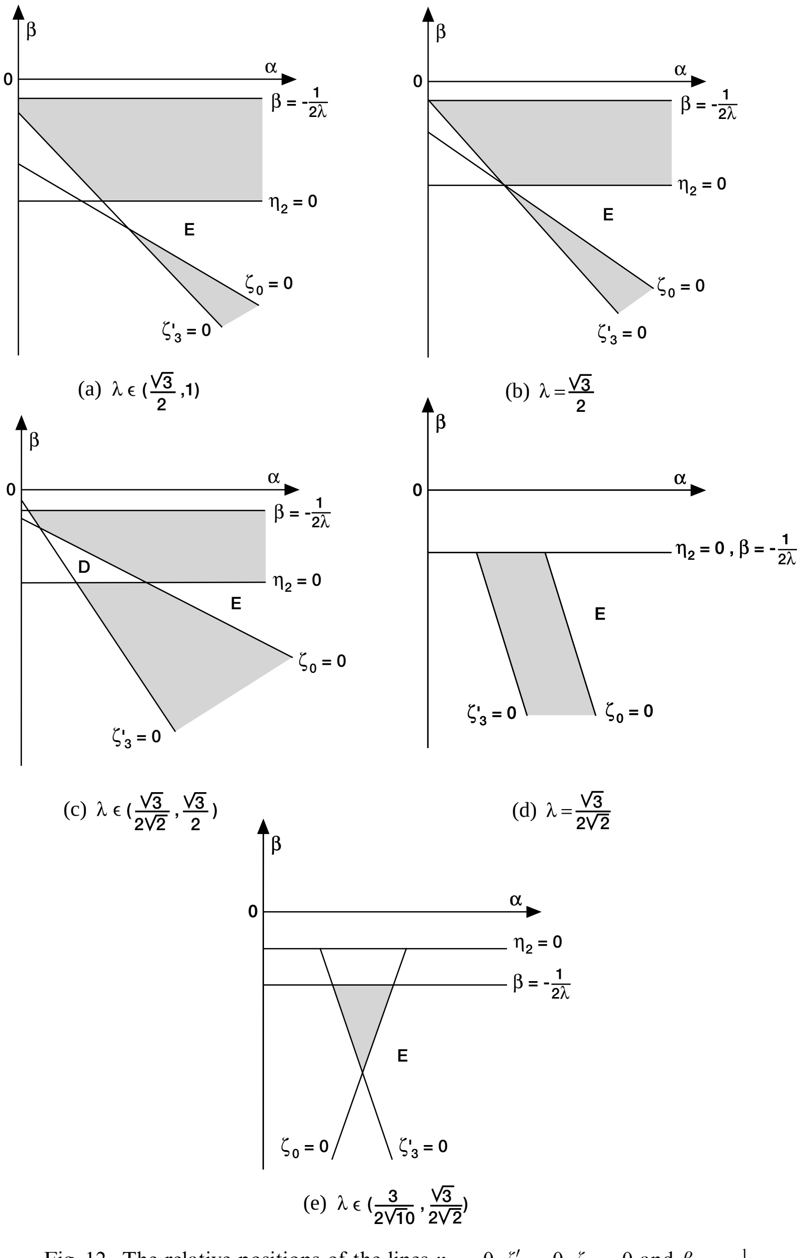

Fig. 12. The relative positions of the lines $\eta_2 = 0$, $\zeta_3' = 0$, $\zeta_0 = 0$ and $\beta = -\frac{1}{2\lambda}$.

Lemma 3.4, $I''(h)$ has at most 3 zeros. Thus, we will suppose

$$3\alpha + 2\lambda\beta + 1 \neq 0, \tag{59}$$

and expression (42) implies that $h = \bar{h}_\lambda$ is not a zero point of $I''(h)$, unless $\bar{h}_\lambda = h^*_\lambda$.

We change (56) into the following form:

$$I''(h) = \frac{18}{\lambda(9 - 8\lambda^2)} I_0''(h)(h - \bar{h}_\lambda)(\omega(h) - U(h)), \tag{60}$$

where

$$U(h) = \frac{24\lambda h - 2(9 - 8\lambda^2)\lambda\alpha}{36h + 2(9 - 8\lambda^2)\lambda\beta + 9 - 8\lambda^2} = \frac{12\lambda h - (9 - 8\lambda^2)\lambda\alpha}{18(h - \bar{h}_\lambda)}. \tag{61}$$

Note that

$$U'(h) = \frac{\lambda(9 - 8\lambda^2)(2\lambda\beta + 3\alpha + 1)}{54(h - \bar{h}_\lambda)^2}, \tag{62}$$

and

$$\lim_{h \to \pm\infty} U(h) = \frac{2\lambda}{3}. \tag{63}$$

Hence the curve C_U, defined by $\omega = U(h)$ in the (h, ω)-plane, consists of two strictly monotone branches C_U^1 and C_U^2 (see condition (59) and expression (62)), and one of them stays above the line $\{\omega = \frac{2\lambda}{3}\}$, while the other is below the same line.

Note also that from system (39) we obtain

$$\dot{\omega} - U'(h)\dot{h}\big|_{\omega = U(h)} = -\frac{\lambda(9 - 8\lambda^2)(n_3 h^3 + n_2 h^2 + n_1 h + n_0)}{648(h - \bar{h}_\lambda)^2}, \tag{64}$$

where the $\{n_i\}$ $(i = 0, \ldots, 3)$ are polynomials in α, β and λ, and

$$n_3 = 1728\zeta_3', \tag{65}$$

where $\zeta_3' = 18\alpha + 12\lambda\beta + 20\lambda^2 - 9$ is the same as before.

As we mentioned above, under condition (59) $h = \bar{h}_\lambda$ is not a zero of $I''(h)$. By (60) the number of zeros of $I''(h)$ is given by the number of intersection points of the curves C_ω and C_U, taking the multiplicity into account. We denote this number by $\#(C_\omega \cap C_U)$.

By Lemma 3.4 for $\lambda \in (0, \frac{3}{2\sqrt{10}}]$ the curve C_ω consists of a unique branch and stays below the line $\{\omega = \frac{2\lambda}{3}\}$, hence it meets at most one branch of C_U. If $\#(C_\omega \cap C_U) \geqslant 5$, then the vector field (39) is tangent to C_U at least at 4 points, which contradicts (64).

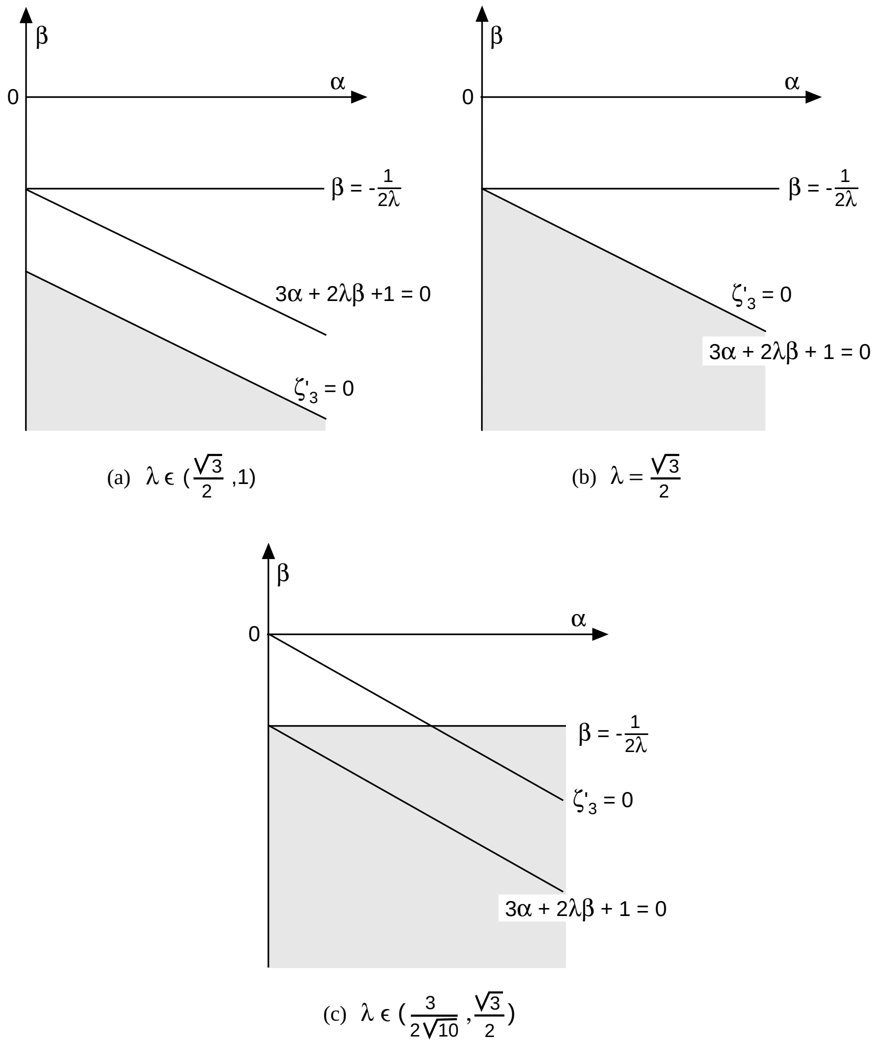

Fig. 13. The relative position of $\zeta_3' = 0$ and $U'(h) = 0$.

For $\lambda \in (\frac{3}{2\sqrt{10}}, 1)$, the case $\zeta_3' \geqslant 0$ was solved in Section 4.1. So we only pay attention to the case $\zeta_3' < 0$. If $\lambda \in [\frac{\sqrt{3}}{2}, 1)$, then we must have $U'(h) < 0$ (see Fig. 13(a) and (b)). If $\lambda \in (\frac{3}{2\sqrt{10}}, \frac{\sqrt{3}}{2})$, then $U'(h)$ may change its sign, see Fig. 13(c).

Lemma 4.7.

(i) $I''(h_\lambda^*) = 0$ *if and only if* $h_\lambda^* = \bar{h}_\lambda$.

(ii) *If* $I^{(k)}(h_\lambda^*) = 0$, $k = 0, 1, 2, 3$, *then* $I^{(4)}(h_\lambda^*) \neq 0$. *In other words, the tangency at the point* $(P, Q)(h_\lambda^*)$ *of the curve* Σ_λ *is at most quadruple.*

Proof. Conclusion (i) follows by (57), (43) and Lemma 3.6. If $I^{(k)}(h_\lambda^*) = 0$, $k = 0, 1, 2, 3, 4$, then by Lemmas 3.6, 3.7 and Remark 3.8, α and β satisfy (45) and (43), respectively, and (λ, h_λ^*) satisfies (46). Besides, by the discussion in Section 4.1, h_λ^* is a triple root of (51), i.e. $J^{(i)}(h_\lambda^*) = 0$, $i = 0, 1, 2$, where $J(h) = \zeta_3 h^3 + \zeta_2 h^2 + \zeta_1 h + \zeta_0$. Substituting (43) and (45) into $J(h_\lambda^*) = 0$, we obtain $K_1(\lambda, h_\lambda^*) = 0$. Eliminating α from $J(h_\lambda^*) = 0$ and $J'(h_\lambda^*) = 0$, and substituting (43) into the resulting equation, we obtain $K_2(\lambda, h_\lambda^*) = 0$. Finally, eliminating h_λ^* from (46) and $K_i(\lambda, h_\lambda^*) = 0$, $i = 1, 2$, we get $L_i(\lambda) = 0$, $i = 1, 2$. Both of them are polynomials of λ. Calculation shows, for example by Maple, $L_1(\lambda) = 0$ and $L_2(\lambda) = 0$ have no common root for $\lambda \in (\frac{3}{2\sqrt{10}}, \frac{\sqrt{3}}{2}) \cup (\frac{\sqrt{3}}{2}, 1)$. If $\lambda = \frac{\sqrt{3}}{2}$, it is easy to find that (46) and $K_1(\frac{\sqrt{3}}{2}, h_\lambda^*) = 0$ have no common root. This finishes the proof of conclusion (ii). $\square$

Denote the two branches of the curve C_U by C_U^1 and C_U^2, respectively, above and below the line $\{\omega = \frac{2\lambda}{3}\}$.

Lemma 4.8. *For* $\zeta_3' < 0$, $\#(C_\omega^2 \cap C_U^1) \leqslant 2$.

Proof. By Lemma 3.4, it is obvious that only in the last case of Fig. 6(c) it is possible to have $C_\omega^2 \cap C_U^1 \neq \phi$, and $\#(C_\omega^2 \cap C_U^1)$ must be even, taking into account the multiplicity. Besides, in this case we must have $\lambda \in (\frac{\sqrt{3}}{2}, 1)$ (Lemma 3.4), implying $U'(h) < 0$, see Fig. 13(a). If $\#(C_\omega^2 \cap C_U^1) > 2$, then $\#(C_\omega^2 \cap C_U^1) \geqslant 4$, and there exist at least 3 tangent points on C_U^1 with respect to the vector field (39) until the most right intersection point M. Near M and right to M, the direction of vector field (39) is pointing downwards to C_U^1 (see Fig. 14).

On the other hand, since $\zeta_3' < 0$, by (64) and(65) for $h \gg 1$ the vector field (39) is pointing upwards with respect to C_U^1, hence there is at least one more tangent point, contradicting (64). $\square$

Proposition 4.9. *If* $\lambda \in (\frac{3}{2\sqrt{10}}, 1)$ *and* $\zeta_3' < 0$, *then* $N(I'') \leqslant 4$, *where* $N(I'')$ *is the number of zeros of* $I''(h)$ *for* $h > 0$.

Proof. By Lemma 4.7(i), if $\bar{h}_\lambda \neq h_\lambda^*$, then $N(I'') = \#(C_\omega \cap C_U)$; if $\bar{h}_\lambda = h_\lambda^*$, then $N(I'') = \#(C_\omega \cap C_U) + M(h_\lambda^*)$, where $M(h_\lambda^*)$ is the multiplicity of h_λ^* as a zero of $I''(h)$. By Lemma 4.7(ii), $M(h_\lambda^*) \leqslant 2$.

Under condition (59), we will consider the two cases of $\chi = 3\alpha + 2\lambda\beta + 1 < 0$ and $\chi > 0$ separately. Note that when $\lambda \in [\frac{\sqrt{3}}{2}, 1)$, $\zeta_3' < 0$ implies $\chi < 0$, see Fig. 13.

From (47), (57) and (61) we have

$$\bar{h}_\lambda > 0, \;\; U(0) = \frac{\lambda(9-8\lambda^2)\alpha}{18\bar{h}_\lambda} > 0 \quad \text{and} \quad U(0) - \frac{2\lambda}{3} = \frac{\lambda(9-8\lambda^2)}{54\bar{h}_\lambda}\chi. \tag{66}$$

We first suppose $\chi < 0$, i.e. $U'(h) < 0$.

If $\bar{h}_\lambda < h^*_\lambda$, then by Lemma 3.4 $\#(C^1_\omega \cap C^1_U) = 1$, by Lemma 4.8, $\#(C^2_\omega \cap C^1_U) \leqslant 2$; and it is obvious that $\#(C_\omega \cap C^2_U) = 0$, see Fig. 15(a). If $\bar{h}_\lambda = h^*_\lambda$, then $\#(C_\omega \cap C_U) = \#(C^2_\omega \cap C^1_U) \leqslant 2$ (Lemma 4.8), and $M(h^*_\lambda) \leqslant 2$ (Lemma 4.7), hence $N(I'') \leqslant 4$, see Fig. 15(b).

If $\bar{h}_\lambda > h^*_\lambda$, then $\#(C_\omega \cap C_U) \leqslant 3$. In fact, if $C^2_\omega \cap C^1_U = \phi$ and $\#(C^2_\omega \cap C^2_U)$ is bigger than 3, it would be at least 5 (it must be odd), resulting in at least 4 "tangent points" on C^2_U, and contradicting (64). If $C^2_\omega \cap C^1_U \neq \phi$, then $\#(C^2_\omega \cap C^1_U) = 2$, resulting in two "tangent points" on C^1_U (see Fig. 15(c) and the analysis in the proof of Lemma 4.8). This implies $\#(C^2_\omega \cap C^2_U) = 1$. In any case $C^1_\omega \cap C_U = \phi$.

In Fig. 15 we only illustrate the case that C^2_ω cuts the line $\{\omega = \frac{2\lambda}{3}\}$. If C^2_ω does not cut this line (see Fig. 6(c)), the proof is basically the same, and the discussion is more simple.

Next, we suppose $\chi > 0$, i.e. $U'(h) > 0$. Then we have $\lambda \in (\frac{3}{2\sqrt{10}}, \frac{\sqrt{3}}{2})$ (see Fig. 13(c)), and C^2_ω does not meet the line $\{\omega = \frac{2\lambda}{3}\}$ (see Lemma 3.4). By (66), $U(0) - \frac{2\lambda}{3} > 0$. We only consider the case $U(0) > \omega(0)$. If $U(0) \leqslant \omega(0)$, then the discussion is similar. The key point is to use the number of tangent points on C_U with respect to the vector field (39) to control $\#(C_\omega \cap C_U)$.

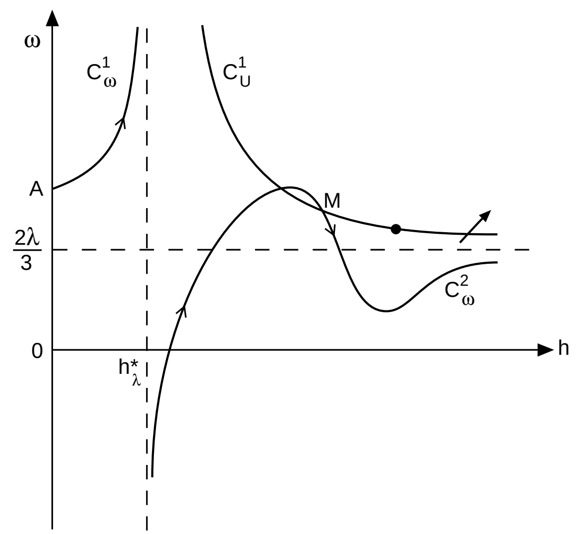

Fig. 14. $C^2_\omega \cap C^1_U \neq \phi$.

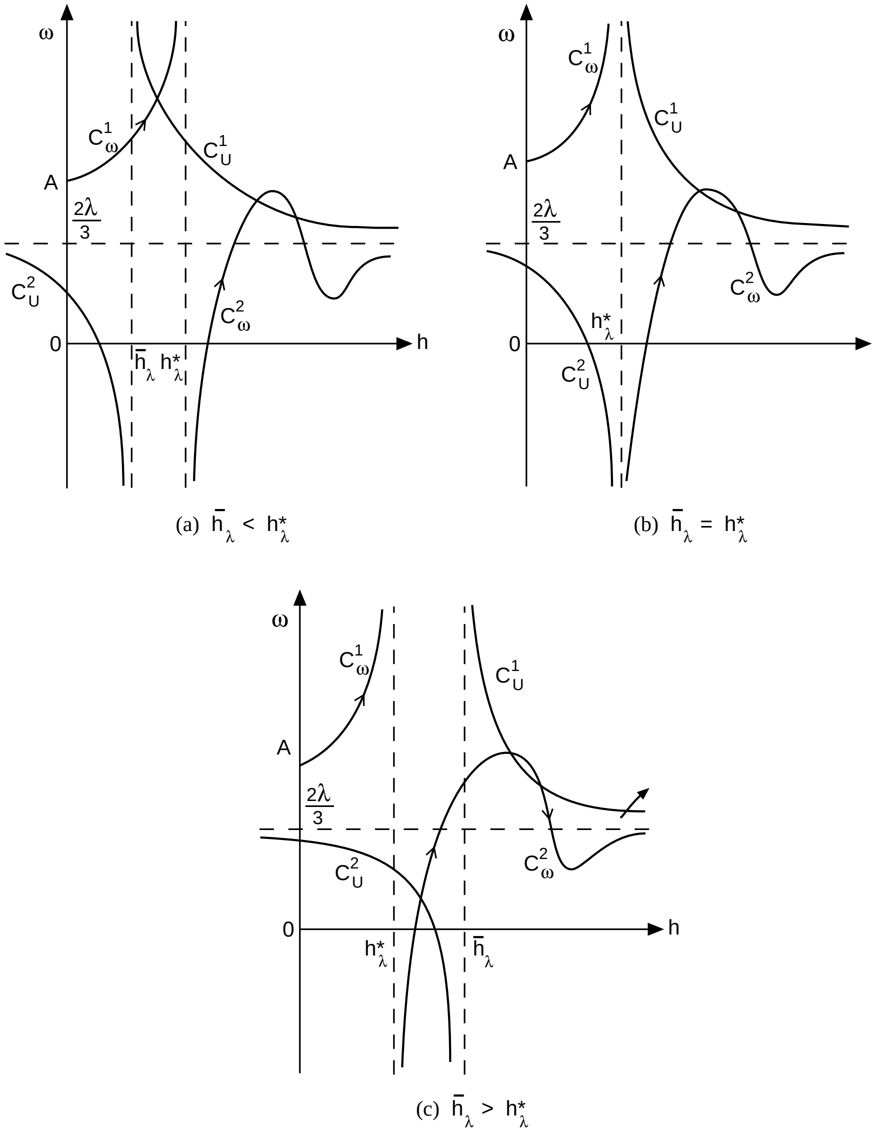

Fig. 15. The case $U'(h) < 0$.

If $\bar{h}_\lambda < h_\lambda^*$, then by the saddle property at the point A, the number of "tangent points" on C_U^1 is equal to $\#(C_\omega^1 \cap C_U^1)$. On the other hand, this number is even, by (64) $\#(C_\omega^1 \cap C_U^1) \leqslant 2$. By (37) and (38) we have $I''(h) = I_0''(h)(\alpha + \beta\omega(h) + v(h))$. As $h \to +\infty$, $I_0''(h) < 0$ (Lemma 3.3(i)), $\omega(h) \to \frac{2\lambda}{3}$ and $v(h) \to -\infty$ (Lemma 3.1(ii)). This implies $I''(h) > 0$ for $h \gg 1$. By (60), C_U must be above C_ω for $h \gg 1$. Hence $\#(C_\omega^2 \cap C_U^2)$ is also even (see Fig. 16(a)). On the other hand, by the same argument as in the proof of Lemma 4.8, $\zeta_3' < 0$ implies one more "tangent point" on C_U^2 right to the most right intersection point of C_ω^2 and C_U^2. Hence $\#(C_U^2 \cap C_\omega^2) \leqslant 2$. By (64) again, $\#(C_\omega^1 \cap C_U^1) = 2$ and $\#(C_\omega^2 \cap C_U^2) = 2$ do not hold at the same time, therefore $\#(C_\omega \cap C_U) \leqslant 2$.

If $\bar{h}_\lambda > h_\lambda^*$, then both $\#(C_\omega^1 \cap C_U^1)$ and $\#(C_\omega^2 \cap C_U^2)$ are odd, and at least 1. By the same arguments, both of them must be 1. Hence $\#(C_\omega \cap C_U) = 2$, see Fig. 16(c).

Finally, if $\bar{h}_\lambda = h_\lambda^*$ and $M(h_\lambda^*) = 1$, then $I''(h)$ changes its sign as h passes the value h_λ^*. By Lemma 3.3(i) and (60), for $0 < |h - h_\lambda^*| \ll 1$, if C_U^1 is above (resp., below) C_ω^1, then C_U^2 must be below (resp., above) C_ω^2. Hence, by the same arguments as above we conclude that if $\#(C_\omega^1 \cap C_U^1) + \#(C_\omega^2 \cap C_U^2) \geqslant 3$ then $(\#(C_\omega^1 \cap C_U^1), \#(C_\omega^2 \cap C_U^2))$ can only take values $(0,3)$, $(1,2)$, $(2,1)$ or $(3,0)$. Therefore $N(I'') = \#(C_\omega \cap C_U) + M(h_\lambda^*) \leqslant 4$. If $\bar{h}_\lambda = h_\lambda^*$ and $M(h_\lambda^*) = 2$, then $I''(h)$ keeps its sign as h passes the value h_λ^*, implying both C_U^1 and C_U^2, respectively, above (or below) C_ω^1 and C_ω^2 at the same time for $0 < |h - h_\lambda^*| \ll 1$. Hence if $\#(C_\omega^1 \cap C_U^1) + \#(C_\omega^2 \cap C_U^2) \geqslant 2$ then $(\#(C_\omega^1 \cap C_U^1), \#(C_\omega^2 \cap C_U^2))$ can only take values $(0,2)$, $(1,1)$ or $(2,0)$. We also obtain $N(I'') \leqslant 4$. $\square$

5. Proof of conclusion (B2)

We use the same method and notation as in Section 4.1. Taking $\lambda \to 0$ in (49) and (51), we obtain

$$f_1(h) = \eta_2 h^2 + \eta_1 h + \eta_0 \to -27\beta(4h+1)(4h+5\alpha)$$

and

$$f_2(h) = \zeta_3 h^3 + \zeta_2 h^2 + \zeta_1 h + \zeta_0 \to -729\beta(4h+1)^2[4(2\alpha-1)h - 15\alpha^2].$$

Note that (α, β) satisfies (47). Hence, for $0 < \alpha \leqslant \frac{1}{4}$, there is a $\delta_1 > 0$, such that for $\lambda \in (0, \delta_1)$ and all $\beta < -\frac{1}{2\lambda}$, $f_1(h)$ has at most one positive zero and $f_2(h)$ has no positive zero; for $\alpha > \frac{1}{4}$, there is a $\delta_2 > 0$, such that for $\lambda \in (0, \delta_2)$ and all $\beta < -\frac{1}{2\lambda}$, $f_1(h)$ has no positive zero and $f_2(h)$ has at most one positive zero. Let $\sigma_1 = \min(\delta_1, \delta_2)$, then by the same discussion as in Section 4.1, for $\lambda \in (0, \sigma_1)$, $\#(C_P \cap C_W) \leqslant 2$, and hence $I(h)$ has at most 4 zeros for $h \geqslant 0$. Note $I(0) = 0$, and by Lemma 2.9 Σ_λ has no triple nor higher than triple point.

6. Proof of conclusion (B3)

Lemma 6.1. $\lim_{h \to 0+} \frac{Q'(h)}{P'(h)} = \frac{1}{2\lambda}$ *and* $\lim_{h \to 0+} \frac{Q''(h)P'(h) - P''(h)Q'(h)}{(P'(h))^3} = \frac{5}{3\lambda^2}$.

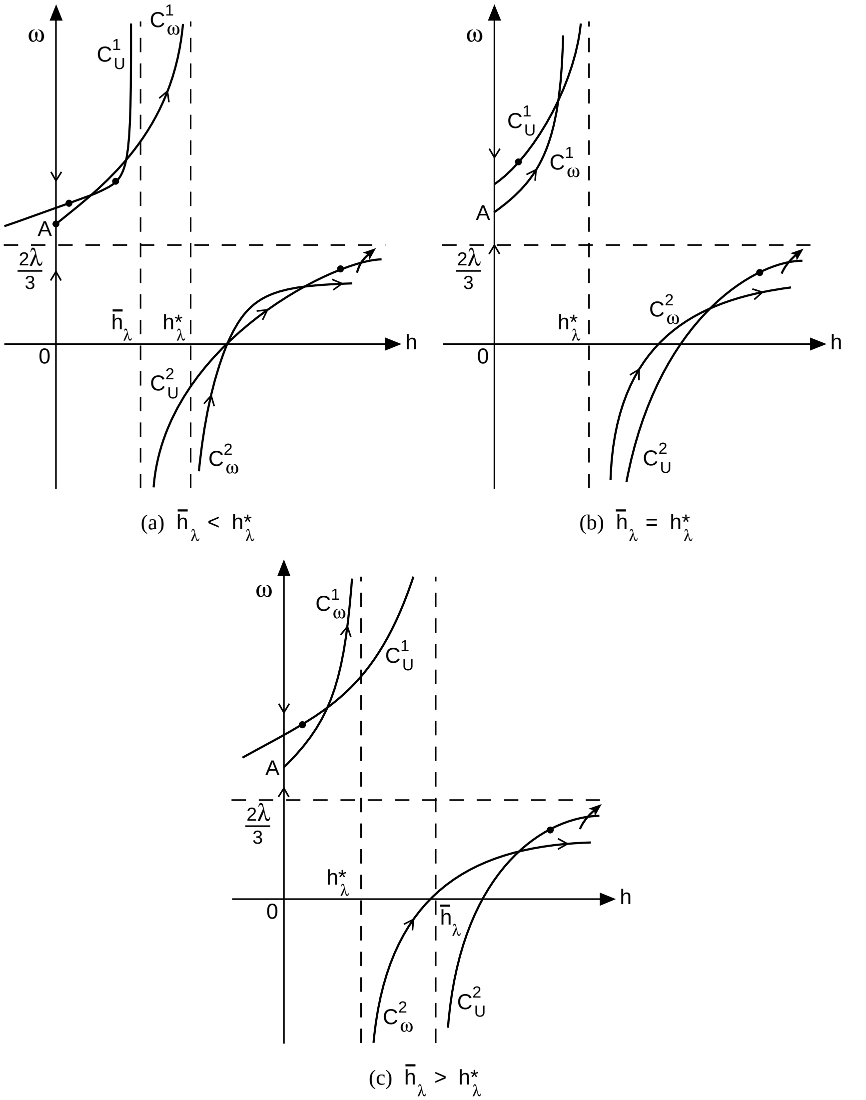

Fig. 16. The case $U'(h) > 0$.

Proof. At the singularity $(h, P, Q) = (0, 0, 0)$ the linearization of system (36) has the matrix

$$12(9 - 8\lambda^2)\begin{pmatrix} 1 & 0 & 0 \\ 2\lambda & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

The orbit under study is the unstable manifold corresponding to the eigenvalue 1 having an eigenvector $\begin{pmatrix} 1 \\ \lambda \\ 1/2 \end{pmatrix}$. Hence $P'(0) = \lambda$, $Q'(0) = \frac{1}{2}$. For $0 < h \ll 1$, let

$$\begin{cases} h = h, \\ P = \lambda h + \dfrac{a_2}{2} h^2 + \cdots, \\ Q = \dfrac{1}{2} h + \dfrac{b_2}{2} h^2 + \cdots. \end{cases}$$

Substituting it into (36) we obtain

$$P''(0) = a_2 = \tfrac{1}{36} \lambda (440\lambda^2 - 183),$$

$$Q''(0) = b_2 = \tfrac{55}{9} \lambda^2 - \tfrac{7}{8}.$$

Lemma 6.1 follows immediately. $\square$

By Lemma 6.1 for $\lambda \in [\sigma_1, 1]$ $\lim_{h \to 0+} \frac{Q''(h)P'(h) - P''(h)Q'(h)}{(P'(h))^3} = \frac{5}{3\lambda^2}$ is strictly positive and decreasing as λ increases from σ_1. Hence, by the smoothness of Σ_λ for $\lambda \in [\sigma_1, 1]$, the fact that the unique non-smooth point of Σ_1 is away from $h = 0$, and by the compactness of the interval $[\sigma_1, 1]$, we can find a positive h_1, independent of λ, such that

$$\frac{Q''(h)P'(h) - P''(h)Q'(h)}{(P'(h))^3} \geqslant \frac{5}{6}$$

for all $h \in [0, h_1]$ and $\lambda \in [\sigma_1, 1]$. Property (B3) of Theorem B is proved.

7. Proof of conclusion (B4)

By using (36) we determine the sign of $\frac{d^2Q}{dP^2}$ along Σ_λ for $h \gg 1$. Since $P'(h) > 0$ for $h > 0$ (Lemma 2.2), we need only to estimate $Q''(h)P'(h) - P''(h)Q'(h)$. Modulo a positive constant it is given by

$$\begin{aligned} \psi(h, \lambda) = {} & 2\left(P(h) - \frac{2\lambda}{3}\right)(18Q(h) - 24\lambda P(h) - 4\lambda^2 + 9)h^3 \\ & + \psi_2(P(h), Q(h), \lambda)h^2 + \psi_1(P(h), Q(h), \lambda)h + \psi_0(P(h), Q(h), \lambda), \end{aligned}$$

where ψ_1 and ψ_0 are polynomials of P, Q and λ, the degree with respect to Q being three, and

$$\psi_2\left(\frac{2\lambda}{3}, Q(h), \lambda\right) = 5\lambda(9-8\lambda^2)(3Q(h)-1)(18Q(h)-20\lambda^2+9).$$

By Lemma 2.7, for $h \gg 1$

$$\begin{aligned}\psi(h,\lambda) &= 72\lambda(9-8\lambda^2)\frac{B\left(\frac{3}{4},\frac{3}{2}\right)}{B\left(\frac{1}{4},\frac{3}{2}\right)}\left[\frac{15B\left(\frac{3}{4},\frac{3}{2}\right)}{B\left(\frac{1}{4},\frac{3}{2}\right)} - \frac{1}{27}\left(1+\frac{16}{B\left(\frac{1}{4},\frac{3}{2}\right)}\right)\right]h^3 \\ &\quad + O(h^{3/2}) \\ &\geqslant 72\sigma_1\left(1+O\left(\frac{1}{h^{1/2}}\right)\right)h^3,\end{aligned}$$

for all $\lambda \in [\sigma_1, 1)$. Hence, there is a $h_2 > h_1$ such that for $h \geqslant h_2$ and all $\lambda \in [\sigma_1, 1)\frac{d^2Q}{dP^2} > 0$ along Σ_λ, i.e. Σ_λ has no triple nor higher than triple point for $h \geqslant h_2$.

8. Proof of conclusion (B5)

Let $\lambda = 1$, then system $(3)_0$ becomes

$$\begin{cases}\dot{x} = y, \\ \dot{y} = -x(x-1)^2.\end{cases} \tag{67}$$

System (67) has a centre at $(0,0)$ and a cuspidal loop related to the singularity at $(1,0)$, corresponding to $h = \frac{1}{12}$. By the appendix of [DL2], the curve Σ_1 has exactly two inflection points M_1 and M_2: M_2 is a triple point and M_1 corresponds to the cuspidal value $h = \frac{1}{12}$, $h|_{M_2} > \frac{1}{12}$, see Fig. 17(a). The tangent line of Σ_1 at M_i must cut Σ_1 at a second point $N_i \in \Sigma_1$, $i = 1,2$ (Lemma 2.9), see Fig. 17(b) and (c).

Note that the right-hand sides of system (36) are polynomials in P, Q, h and λ; $\Sigma_\lambda \in C^\infty(0,+\infty)$ for $\lambda \in (0,1)$ and $\Sigma_1 \in C^1(0,+\infty) \cap C^\infty((0,\frac{1}{12}) \cup (\frac{1}{12},+\infty))$. Let us take sufficiently small neighbourhoods $U^{(i)} \subset \Sigma_1$ of the respective points M_i, $i = 1,2$. Then there is a small $\sigma_2 > 0$, such that for $\lambda \in (1-\sigma_2, 1)$ we have:

(i) Σ_λ has no triple nor higher than triple points for $h \in [h_1, h_2]$ and outside $U_\lambda^{(i)} \subset \Sigma_\lambda$, where the $U_\lambda^{(i)}$ are continuous deformations of $U^{(i)}$, $i = 1,2$;

(ii) at the end points of $U_\lambda^{(i)}$ $(i = 1,2)$ on Σ_λ, $\frac{d^2Q}{dP^2}$ has different signs; and

(iii) at any point on $U_\lambda^{(i)}$, the tangent line of Σ_λ cuts Σ_λ at a point near $N_{\lambda i}$, the continuous deformation of N_i, $i = 1,2$ (cf. Fig. 17(b) and (c)).

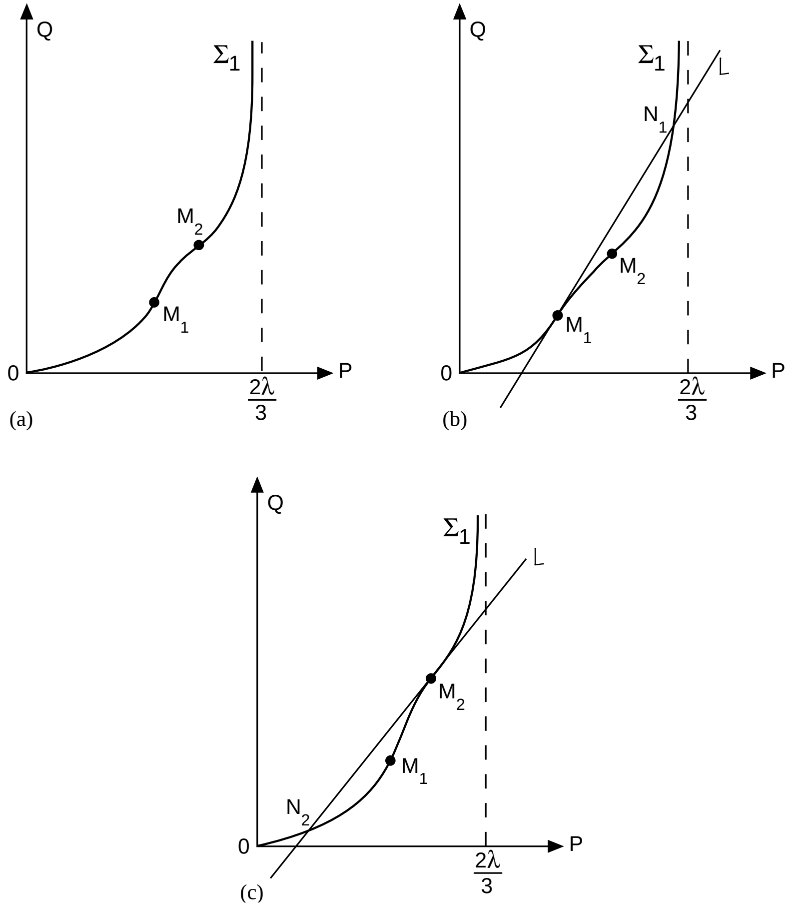

Fig. 17. The behaviour of Σ_1.

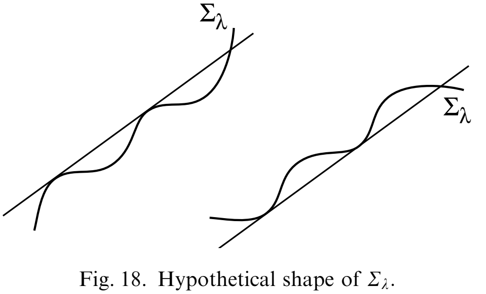

Fig. 18. Hypothetical shape of Σ_λ.

Let us first show that in each $U_\lambda^{(i)}$ there is no quadruple nor higher than quadruple point. In fact, the tangent line at such a point, by property (iii) above, must cut at least one more point on Σ_λ. This contradicts conclusion (B1) of Theorem B.

By property (ii), in each $U_\lambda^{(i)}$ there exists at least one triple point, $i = 1, 2$. Let us show that the triple point is unique. In fact, if this was not the case, then there would at least be three triple points in a $U_\lambda^{(i)}$, inducing the existence of a straight line, tangent to Σ_λ at two points on $U_\lambda^{(i)}$ at the same time (see Fig. 18). By property (iii), this also contradicts conclusion (B1) of Theorem B. $\square$

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